Introduction:

If you recall The fundamental Theorem of calculus, we had

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$, where F' is a continuous function on $[a,b]$. The above equation was also called **Net change Theorem.**

* If we think of gradient vector ∇f of a multivariable function as a sort of derivative of f, then the following theorem can be regarded as a version of the fundamental theorem for line integrals.

Theorem: Let C be a smooth curve given by the vector function $r(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. then:

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Note: this theorem says that we can evaluate the line integral of a conservative vector field (gradient vector field of the potential function f) simply by knowing the value of f at the endpoints (initial point and terminal point of curve C)

Note: This Theorem also says that the line integral of ∇f is the net change in f. e.g. for a 2D curve if the initial point is $A(x_1, y)$ and the end point is $B(x_2, y_2)$ then $\int_C \vec{\nabla} f \cdot \vec{dr} = \int \vec{F} \cdot \vec{dr} = f(x_2, y_2) - f(x_1, y_1)$

Now, if f is a function of three variables and C is a space curve from initial point $A(x_1, y_1, z_1)$ and endpoint $B(x_2, y_2, z)$, regardless of the path

$$\int_C \vec{\nabla} f \cdot \vec{dr} = \int_C \vec{F} \cdot \vec{dr} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

The proof for this theorem is very simple:

Suppose that we have a conservative vector field $\vec{F} = \nabla f$

$$\Rightarrow \int_c F \cdot dr = \int_c \nabla f \cdot dr = \int_a^b \nabla f(r(t)) \cdot r'(t)\,dt$$

This really means $\int_a^b F(r(t)) \cdot r'(t)\,dt$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt$$

$$= \int_a^b \underbrace{\left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right)} \cdot dt$$

This can be written as $\frac{d}{dt}[f(x(t), y(t), z(t)]$ (By the chain rule)

$$= \int_a^b \frac{d}{dt}[f(x(t), y(t), z(t)] \cdot dt = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$= f(r(b)) - f(r(a))$$

* **Note:** This means if we calculate line integral of a conservative field over a closed path $\Rightarrow \int_c F \cdot dr = 0$ because the initial point and the terminal points are the same

$$f(r(a)) - f(r(a)) = 0$$

Theorem: if $F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on domain D, then throughout D we have:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$ → very important

Note (important): Although, the converse of this theorem is not always correct, since the domain that we define for our vector field is almost always $\mathbb{R}^2$ or $\mathbb{R}^3$, assuming that we have continuous first order partial derivatives for P and Q (in $\mathbb{R}^3$ also for R), we may assume that the converse of this theorem is always correct.

Therefore in 2D cases where $F = P\vec{i} + Q\vec{j}$

if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then F is conservative

Therefore in 3D cases where $F = P\vec{i} + Q\vec{j} + R\vec{k}$

if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, and $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ then $\vec{F}$ is conservative.

Example: Determine whether or not the following vector field is conservative

$F(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$

solution: lets call $P(x,y) = x-y$ and $Q(x,y) = x-2$

$$\frac{\partial P}{\partial y} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1 \quad \Longrightarrow \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

Therefore, F is not conservative.

Example: Determine whether or not the vector field $F(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ is conservative.

solution: $P(x,y) = (3+2xy)$, $Q(x,y) = +(x^2-3y^2)$

$$\frac{\partial P}{\partial y} = 2x \quad , \quad \frac{\partial Q}{\partial x} = 2x \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow F \text{ is conservative.}$$

Example: (important) (a) if $F(x,y) = \underbrace{(3+2xy)}_{P}\vec{i} + \underbrace{(x^2-3y^2)}_{Q}\vec{j}$ find the potential function such that $F = \nabla f$

(b) Evaluate $\int_C F.dr$ where C is the curve given by $r(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j}$
$0 \le t \le \pi$

Solution: from the previous problem, we know that F is a conservative field.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow \frac{\partial f}{\partial x} = 3+2xy \Rightarrow f(x,y) = \int (3+2xy)\,dx = 3x + x^2y + g(y) \\ &\frac{\partial f}{\partial y} = x^2 - 3y^2 = x^2 + g'(y) \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + k \end{aligned}$$

$$\Rightarrow f(x,y) = 3x + x^2y - y^3 + k$$

Part b) instead of going the long way to find $\int_C F.dr$, we can use fundamental theorem of line integrals: The end points are at $t=0$ and $t=\pi$

$$r(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j} \Rightarrow \begin{cases} x(t) = e^t \sin t \\ y(t) = e^t \cos t \end{cases}$$

$$t=0 \Rightarrow \begin{aligned} x(0) &= e^0 \sin 0 = 0 \\ y(0) &= e^0 \cos 0 = 1 \end{aligned} \quad (x,y) = (0,1)$$

$$t=\pi \Rightarrow \begin{aligned} x(\pi) &= e^\pi \sin \pi = 0 \\ y(\pi) &= e^\pi \cos \pi = -e^\pi \end{aligned} \quad (x,y) = (0,-e^\pi)$$

$$\Rightarrow \int_C F.dr = \int \nabla f.dr = f(0,-e^\pi) - f(0,1) = e^{3\pi} - (-1) = e^{3\pi} + 1$$

Example: if $F(x,y,z) = \underbrace{y^2}_{P}\vec{i} + \underbrace{(2xy + e^{3z})}_{Q}\vec{j} + \underbrace{3ye}_{R}{}^{3z}\vec{k}$, find a function such that $\nabla f = F$.

Solution: if F is a Conservative field $\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$

$\dfrac{\partial P}{\partial y} = 2y$ $\qquad$ $\dfrac{\partial P}{\partial z} = 0$ $\qquad$ $\dfrac{\partial Q}{\partial z} = 3e^{3z}$

✓ $\qquad$ ✓ $\qquad$ ✓

$\dfrac{\partial Q}{\partial x} = 2y$ $\qquad$ $\dfrac{\partial R}{\partial x} = 0$ $\qquad$ $\dfrac{\partial R}{\partial y} = 3e^{3z}$

So we can say that F is conservative

$$P = \frac{\partial f}{\partial x} = y^2 \Rightarrow f(x,y,z) = \int y^2 \cdot dx = xy^2 + g(y,z)$$

$$Q = \frac{\partial f}{\partial y} = 2xy + e^{3z} = \frac{\partial}{\partial y}\left(xy^2 + g(y,z)\right) = 2xy + g_y(y,z)$$

$$\Rightarrow g_y(y,z) = e^{3z} \Rightarrow g(y,z) = \int e^{3z}\,dy = e^{3z}\cdot y + h(z)$$

$$\Rightarrow f(x,y,z) = xy^2 + ye^{3z} + h(z)$$

$$\frac{\partial f}{\partial z} = R = 3ye^{3z} = \frac{\partial}{\partial z}\left(xy^2 + ye^{3z} + h(z)\right) = 0 + 3ye^{3z} + h'(z)$$

$$\Rightarrow h'(z) = 0$$

$$\Rightarrow h(z) = C$$

$$\Rightarrow f(x,y,z) = xy^2 + ye^{3z} + C$$

Example: find the work done by $\vec{F}(x,y) = (x + \tan^{-1} y)\vec{i} + \left(\frac{x+y}{1+y^2}\right)\vec{j}$ to move a particle from Point $A(x_1, y_1)$ to Point $B(x_2, y_2)$.

Solution: First, let's see if this field is Conservative.

$P(x,y) = x + \tan^{-1} y$ is $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$?

$Q(x,y) = \frac{x+y}{1+y^2}$

$\frac{\partial P}{\partial y} = \frac{1}{1+y^2}$, $\frac{\partial Q}{\partial x} = \frac{1}{1+x^2}$ ✓

Therefore F is a Conservative Field

so $\nabla f = F$

$$P(x,y) = \frac{\partial f(x,y)}{\partial x} \Rightarrow f(x,y) = \int (x + \tan^{-1} y)\, dx = \frac{x^2}{2} + x \tan^{-1} y + g(y)$$

$$Q(x,y) = \frac{\partial f(x,y)}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{x^2}{2} + x\tan^{-1} y + g(y)\right) = \frac{x}{1+y^2} + g'(y) = \frac{x+y}{1+y^2}$$

$$\Rightarrow g'(y) = \frac{y}{1+y^2} \Rightarrow g(y) = \int \frac{y}{1+y^2}\, dy$$

$$g(y) = \tfrac{1}{2} \ln(1+y^2)$$

$$\Rightarrow f(x,y) = \frac{x^2}{2} + x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C$$