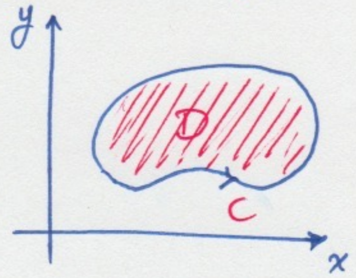
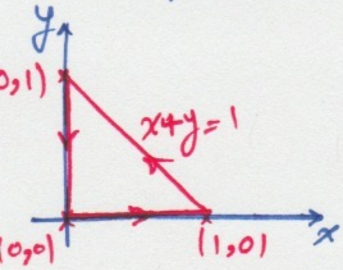
Introduction: Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.



> **Green's Theorem:** let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, Then:
>
> $$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example: Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segment from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution: One way to solve this problem is to separate this closed path into three different line segments, find the parametric eq. of line segments and then evaluate the above line integral along each of those parametric paths (section 16.2). However, we can use Green's Theorem for this problem:



According to Green's Theorem

$$\oint P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \begin{cases} P = x^4 \\ Q = xy \end{cases}$$

$$= \iint_D (y - 0)\,dA = \int_0^1 \int_0^{1-x} y \cdot dy\,dx$$

$$= \tfrac{1}{2}\int_0^1 y^2\Big]_0^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{6}(1-x)^3\Big]_0^1$$

$$= -\tfrac{1}{6}(-1) = \boxed{\tfrac{1}{6}}$$

Example: Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2+y^2=9$.

Solution:
$$P(x,y) = (3y - e^{\sin x})$$
$$Q(x,y) = (7x + \sqrt{y^4+1})$$

Green's Theorem: $\oint P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

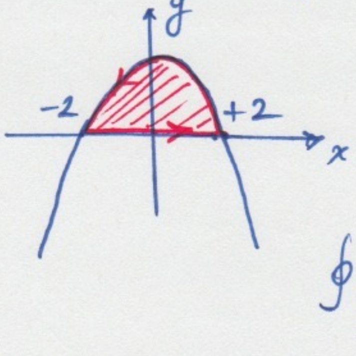
$$\Rightarrow \quad = \iint_D (7-3)\,dA = 4\iint_D dA = 4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

we convert everything to polar coordinates.

$$= 4\int_0^{2\pi} \left.\frac{r^2}{2}\right]_0^3 d\theta = 4\int_0^{2\pi} \frac{9}{2}\,d\theta$$

$$= 18\int_0^{2\pi} d\theta = \boxed{36\pi}$$

Example: Evaluate $\oint (y^2 + \cos x)\,dx + (x - \text{Arctan}\, y)\,dy$ along the simple closed curve given by region bounded by the curves $y = 4 - x^2$ and $y = 0$.



$$P = y^2 + \cos x \qquad Q = x - \text{Arctan}\, y$$

$$\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = 1$$

$$\oint_C F\cdot dr = \iint_D (1-2y)\,dA = \int_{-2}^{2}\int_0^{4-x^2} (1-2y)\,dy\,dx$$

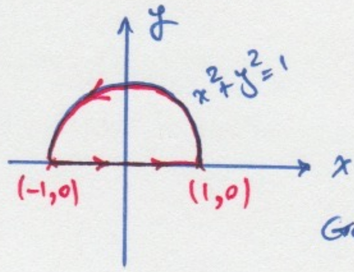
$$= \int_{-2}^{2} \left. y - y^2 \right]_0^{4-x^2} dx = \int_{-2}^{2} \left[(4-x^2) - (4-x^2)^2\right] dx$$

After simplification $\longrightarrow$ $= \int_{-2}^{2} \underbrace{(7x^2 - x^4 - 12)}_{\text{even func.}}\,dx = 2\int_0^2 (7x^2 - x^4 - 12)\,dx$

Final Ans. $= \frac{-352}{15}$

Example: Evaluate $\oint x^2y\,dx + y^3\,dy$ along the path that starts at Point $(-1,0)$ and goes to the point $(1,0)$ and then goes back to the starting point following $x^2+y^2=1$.

Solution:



$P = x^2y \qquad \frac{\partial Q}{\partial x} = 0$

$Q = y^3 \qquad \frac{\partial P}{\partial y} = x^2$

Green: $\oint x^2y\,dx + y^3\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (-x^2)\,dA$

$= -\int_0^{\pi}\int_0^1 (\overset{r\cos\theta}{x^2})\, r\,dr\,d\theta = -\int_0^{\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$

$= -\frac{1}{4}\int_0^{\pi} r^4\cos^2\theta\Big]_0^1 d\theta = -\frac{1}{4}\int_0^{\pi} \frac{1+\cos 2\theta}{2}\,d\theta$

$= -\frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi} = -\frac{1}{8}(\pi) = -\frac{\pi}{8}$

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let's have a new idea about area calculation:

We know according to green's Theorem: $\oint P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

we know if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, $\text{Area} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\Rightarrow \text{Area} = \frac{1}{2}\iint_D 2\,dA = \frac{1}{2}\iint [(1)-(-1)]\,dA$. This means if we find P and Q such that $\frac{\partial P}{\partial y} = -1$, and $\frac{\partial Q}{\partial x} = 1$, Then $\frac{1}{2}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ will give us the area of region D.

in other words $\frac{1}{2}\oint_C P\,dx + Q\,dy = \text{Area}$ for such choice of P and Q.

$\frac{\partial Q}{\partial x} = 1 \Rightarrow Q(x,y) = x \qquad \frac{\partial P}{\partial y} = -1 \Rightarrow P(x,y) = -y$

Therefore, if $Q(x,y)=x$ and $P(x,y)=-y$, $\frac{1}{2}\oint_C -y\,dx + x\,dy$ gives us the area of the region D enclosed by curve C.

*Note: remember that the choice of P and Q for this purpose is not unique

other choices (e.g.) $\begin{cases} P(x,y)=0 \\ Q(x,y)=x \end{cases}$ $\quad$ $\begin{matrix} P(x,y)=-y \\ Q(x,y)=0 \end{matrix}$

So we may say $$A=\oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint -y\,dx + x\,dy$$

Application: Sometimes parametrizing a curve (closed) and solving the line integral is simple so we take above approach to calculate the area of geometric shapes.

Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution: we know that the parametric equation for an ellipse is:

$$\begin{cases} x=a\cos t \\ y=b\sin t \end{cases} \quad 0\le t\le 2\pi \qquad \Rightarrow \begin{cases} dx=-a\sin t.dt \\ dy=b\cos t.dt \end{cases}$$

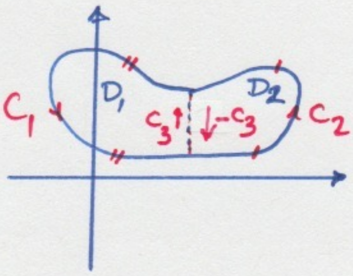
we know $A=\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -b\sin t\,(-a\sin t\,dt) + b\sin t\,(b\cos t)$

$$=\frac{1}{2}\int_0^{2\pi}\left[ab\sin^2 t + ab\cos^2 t\right]dt = \frac{1}{2}\int_0^{2\pi} ab\,dt$$

$$=\frac{1}{2}ab\,t\Big]_0^{2\pi} = \pi ab$$ Area of an ellipse

Extended versions of Green's Theorem:

Sofar, we have used Green's Theorem for the cases that region D is simple. However, we can extend the theorem to the cases where D is a finite union of simple regions.



$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_1 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\Rightarrow \int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where $D = D_1 \cup D_2$

Example: Evaluate $\oint_C y^2 dx + 3xy\,dy$ where C is the boundary of the Semiannular region D in the upper half-plane b/w circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Solution: Notice that y-axis divides the region into two simple regions and we can directly apply the Green's Theorem.

$$\oint y^2 dx + 3xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (3y - 2y)\,dA = \iint_D y\,dA$$

$$= \int_0^{\pi} \int_1^2 (r \sin\theta) r\,dr\,d\theta = \int_0^{\pi} \int_1^2 r^2 \sin\theta\,dr\,d\theta = \int_0^{\pi} \frac{r^3}{3} \sin\theta \Big]_1^2$$

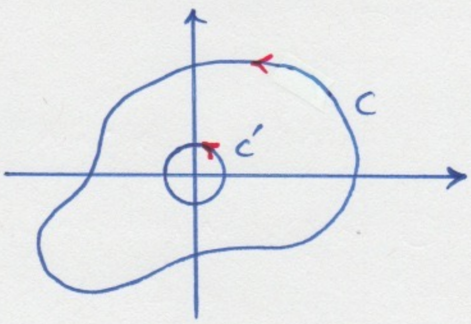
$$= \int_0^{\pi} \left(\frac{8}{3} - \frac{1}{3} \right) \sin\theta\,d\theta = \frac{7}{3} \cdot -(\cos\theta) \Big]_0^{\pi} = -\frac{7}{3} (\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1})$$

$$= -\frac{7}{3}(-2) = +\frac{14}{3}$$

Note: Green's theorem can be extended to regions with holes, that is, regions that are not simply-connected.

Example: if $F(x,y) = (-y\vec{i} + x\vec{j}) / x^2+y^2$, show that $\int_C F.dr = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Solution: Since we have an arbitrary closed path that enclosed the origin, it is difficult to compute the given line integral directly.
Therefore, let's consider a counter clockwise-oriented circle C' with the center at origin and radius of a. a is chosen to be small enough that C' lies inside C.



$$\int_C P dx + Q dy + \int_{-C'} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) A$$

$$\frac{\partial Q}{\partial x} = \frac{1(x^2+y^2) - 2x(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-1(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \int_C P dx + Q dy + \int_{C'} P dx + Q dy = 0$$

$$\Rightarrow \int_C P dx + Q dy = \int_{C'} P dx + Q dy$$

$$\int_C F.dr = \int_{C'} F.dr = \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + a\cos t \cdot (a\cos t)}{a^2\cos^2 t + a^2\sin^2 t} . dt$$

$$= \int_0^{2\pi} \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2\sin^2 t + a^2\cos^2 t} dt = \int_0^{2\pi} dt = 2\pi$$

Therefore $\int_C P dx + Q dy = 2\pi$ for any arbitrary closed path.