Curl:

if $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of F is the vector field on $\mathbb{R}^3$ defined by:

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

As an aid to memory, you can think of the following operator called "del", ∇ as:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Therefore, the easiest way to remember the above definition is by the means of the symbolic expression $\text{Curl } F = \nabla \times F$

Example: If $F(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$, find Curl F.

Solution: $$\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\vec{i} - \vec{j}(0 - x) + (yz - 0)\vec{k}$$

$$\Rightarrow \text{Curl } F = -y(2+x)\vec{i} + x\vec{j} + yz\vec{k}$$

Note: Remember that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$. Therefore, we can compute its curl.

Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then:

$$\boxed{\text{Curl}(\nabla f) = 0 \quad \text{that is} \quad \nabla \times \nabla f = 0}$$

Proof: (you need to know the proof of these Theorems)

$$\text{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{K}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$\Rightarrow$ If F is Conservative, Then Curl $F = 0$

Note: This Theorem gives us a way to check whether or not a vector field is conservative.

Example: show that the vector field $F(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$ is <u>not</u> conservative.

Solution: this is the Same vector field as in example 1/.

we showed that $\text{curl } F = -y(2+x)\vec{i} + x\vec{j} + yz\vec{k}$

$\Rightarrow \text{curl } F \neq 0$ so F is not a conservative field.

Theorem: If F is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $F = 0$, then F is a conservative vector field.

Example: a) Show that $F(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2 k$ is a conservative vector field.

b) Find a function f such that $F = \nabla f$

Solution: for part a), first we calculate $\nabla \times F$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$\text{Curl } F = (\cancel{6xyz^2} - \cancel{6xyz^2})\vec{i} - (\cancel{3y^2z^2} - \cancel{3y^2z^2})\vec{j} + (\cancel{2yz^2} - \cancel{2yz^2})\vec{k}$$

Curl $F = 0 \Rightarrow$ **F is a conservative field.**

Part b) we know $f_x = y^2z^3$, $f_y = 2xyz^2$, $f_z = 3xy^2z^2$

$$\Rightarrow f(x,y,z) = \int y^2z^3\,dx = xy^2z^3 + g(y,z)$$

This is the part that has been cancelled when taken derivative wrt x.

$$f_y(x,y,z) = 2xyz^3 = \frac{\partial}{\partial y}\left(xy^2z^3 + g(y,z)\right) = 2xyz^3 + g_y(y,z)$$

$$\Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

Since the derivative of g wrt to y is zero, this means that g is only a function of z.

$$f_z(x,y,z) = 3xy^2z^2 = \frac{\partial}{\partial z}\left(xy^2z^3 + h(z)\right) = 3xy^2z^2 + h'(z)$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = k \text{ constant}$$

$$\Rightarrow f(x,y,z) = xy^2z^3 + k$$

Divergence:

If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, Then The divergence of F is a function of three variables defined by:

$$\boxed{\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Again, as a memory aid we can write down the divergence of a vector field as the dot product of operator ∇ and the vector field.

$$\nabla \cdot F = \text{Div } F$$

Example: if $F(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\vec{k}$, find div of F.

Solution: $\text{div } F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

Note: if F is a vector field, Then Curl F is also a vector field. Therefore, we can Compute the div of curl of a vector field.

Theorem: If $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second order Partial derivatives, then:

$$\text{div curl } F = 0 \quad \text{That is} \quad \nabla \cdot (\nabla \times F) = 0$$

Proof: $\nabla\cdot(\nabla\times F)$ may be written as

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 R}{\partial x\partial y}} - \cancel{\frac{\partial^2 Q}{\partial x\partial z}} + \cancel{\frac{\partial^2 P}{\partial y\partial z}} - \cancel{\frac{\partial^2 R}{\partial y\partial x}} + \cancel{\frac{\partial^2 Q}{\partial z\partial x}} - \cancel{\frac{\partial^2 P}{\partial z\partial y}} = 0$$

$$\Longrightarrow \nabla\cdot(\nabla\times F)=0$$

Example: show that the vector field $F(x,y,z)=xz\vec{i}+xyz\vec{j}-y^2\vec{k}$ can't be written as the curl of another vector field. That is, $F\neq \text{curl}\, G$

Solution: in the previous example we showed div $F = z + xz$

therefore dif $F\neq 0$, we also know (by the Theorem)

div $F=0$ if $F=$ Curl $G \longrightarrow$ because $\nabla\cdot(\nabla\times G)=0$

therefore div (curl G) $=0$ = div (F) however we saw that div $F\neq 0$ therefore we can't write F as the curl of another vector field like G.

Meaning of the divergence:

Think about it this way, if $F(x,y,z)$ is the velocity of the fluid (or gas), then div $F(x,y,z)$ represents the net rate of change (wrt time) of the mass of fluid flowing from the point (x,y,z) per unit volume.

In other words, div $F(x,y,z)$ measures the tendency of the fluid to diverge from the point (x,y,z).

Laplace operator:

Another differential operator occurs when we compute the divergence of gradient vector field ∇f.

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \text{ which we show as } \nabla^2 f$$

This is called Laplace operator.

we can apply the Laplace operator to a vector field as well:

$$F = P\vec{i} + Q\vec{j} + R\vec{k} \implies \nabla^2 F = \nabla^2 P\,\vec{i} + \nabla^2 Q\,\vec{j} + \nabla^2 R\,\vec{k}$$

Vector Forms of Green's Theorem:

1) Curl and divergence operators, allow us to rewrite Green's Theorem.

suppose that we have a vector field $F = P\vec{i} + Q\vec{j}$ over closed curve C.

we know $\oint_C \vec{F}\cdot\vec{t}\,ds = \oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

Now, let's Calculate Curl F.

$$\operatorname{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\implies \operatorname{Curl} F\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

we can rewrite the above integral as

$$\oint F\cdot dr = \oint P\,dx + Q\,dy = \iint_D (\operatorname{curl} F)\cdot\vec{k}\,dA$$

which says, the line integral of the tangential component of F along closed curve C can be written as the double integral of the vertical component of Curl F over the region D enclosed by C.

Similarly, we can rewrite the line integral of the Normal Component of F along closed curve C as followed:

if c is the curve given by the vector equation

$$r(t) = x(t)\,i + y(t)\,j \qquad a \le t \le b$$

Then unit tangent vector : $T(t) = \frac{x'(t)}{|r'(t)|}\vec{i} + \frac{y'(t)}{|r'(t)|}\vec{j}$

Therefore, the outward unit normal vector to C is :

$$n(t) = \frac{y'(t)}{|r'(t)|}\vec{i} - \frac{x'(t)}{|r'(t)|}\vec{j}$$

You can check $T(t) \cdot n(t) = 0$

$$\Rightarrow \oint_C \vec{F} \cdot \vec{n(t)}\, ds = \int_a^b (F \cdot n)\, |r'(t)|\, dt \qquad ds = |r'(t)|\, dt$$

$$= \int_a^b \left(P\vec{i} + Q\vec{j}\right) \cdot \left(\frac{y'(t)}{|r'(t)|}\vec{i} - \frac{x'(t)}{|r'(t)|}\vec{j}\right) |r'(t)| \cdot dt$$

$$= \int_a^b P \cdot \underbrace{y'(t)\,dt}_{dy} - Q\,\underbrace{x'(t)\,dt}_{dx} = \oint_C P\,dy - Q\,dx$$

Coefficient of dx

Coefficient of dy

$$\Rightarrow \oint_C -Q\,dx + P\,dy \xrightarrow[\text{Theorem}]{\text{Green's}} = \iint_D \left[\frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y}\right)\right] dA$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \operatorname{div} F \cdot dA$$

This version of Green's Theorem says that the line integral of The normal component of F along C is equal to The double integral of the divergence of F over the region D enclosed by C.