Parametric surfaces:

Similar to the idea of parametric curves, sometimes it is easier to represent a surface such as $f(x,y,z)=C$ by the means of a position vector (a vector such that its initial point is the origin of the Coordinate system and its tip or end point is one of the points on the surface). In order to write the Parametric eq. of the surface, we should pick u, and v (as opposed to t for parametric curves), such that

$$\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right\}$$ in other words, we need to write x, y, and z in terms of parameters u, and v.

Then the surface may be shown by $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

Example: Find the parametric representation of the surface $x = \sqrt{y^2+z^2}$

Note: if the eq. of the surface is given in terms of two other variables, just let one be u and the other be v and use u and v to rewrite the third variables. (This is called Trivial paramtrization)

Solution: let $y = u$ and $z = v$ $\Rightarrow x = \sqrt{u^2+v^2}$

$$\Rightarrow \vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$$\Rightarrow \vec{r}(u,v) = \sqrt{u^2+v^2}\,\vec{i} + u\vec{j} + v\vec{k}$$

Example: find the parametric representation of surface with the following eq. $x^2+y^2=z^2$

Solution: Taking the square root makes this problem even more complicated, however, using a simple parametrization (similar to cylindrical coordinates) will easily do the trick

let's assume $\begin{cases} x(u,v) = u\cos v \\ y(u,v) = u\sin v \\ z(u,v) = u \end{cases} \Rightarrow \vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}$

This is the parametric rep. of a cone that its height is along z axis.

Example: Find the parametric representation of the sphere $x^2+y^2+z^2=a^2$

Solution: for this problem, we can get help from spherical coordinate system:

$x(u,v) = a\sin u\cos v$

$y(u,v) = a\sin u\sin v$

$z(u,v) = a\cos u$

$$\Rightarrow \vec{r}(u,v) = a\sin u\cos v\,\vec{i} + a\sin u\sin v\,\vec{j} + a\cos u\,\vec{k}$$

$0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$

Example: Find the parametric representation for the cylinder

$x^2+y^2=4 \quad 0 \leq z \leq 1$

Solution: $\left.\begin{array}{l} x = 2\cos u \\ y = 2\sin u \\ z = v \end{array}\right\} \Rightarrow \vec{r}(u,v) = 2\cos u\,\vec{i} + 2\sin u\,\vec{j} + v\,\vec{k}$

$0 \leq u \leq 2\pi$

$0 \leq v \leq 1$

Example: find a vector function that represent the elliptic paraboloid $z = x^2 + 2y^2$.

Solution: we can use the trivial Parametrization.

$$\left.\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= 2v^2 + u^2 \end{aligned}\right\} \Rightarrow \vec{r}(u,v) = u\vec{i} + v\vec{j} + (u^2+2v^2)\vec{k}$$

Example: Find a parametric representation for the surface $z = 2\sqrt{x^2+y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

Solution 1/ $x(u,v) = u$, $y(u,v) = v \Rightarrow z = 2\sqrt{u^2+v^2}$ (Trivial)

$$\Rightarrow \vec{r}(u,v) = u\vec{i} + v\vec{j} + 2\sqrt{u^2+v^2}\,\vec{k}$$

Solution 2/ $x(u,v) = u\cos v$, $y(u,v) = u\sin v$, $z = 2\sqrt{u^2\cos^2 v + u^2\sin^2 v}$

, $z = 2\sqrt{u^2} = 2u$

$$\Rightarrow \vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + 2u\vec{k}$$

Reasons For Parametrization:

why do we care about having the surfaces Parametrized?

1- instead of 3 variables, we deal w/ 2

2- when we have $r(u,v)$, we can easily calculate the area of a surface:

Cross product

Derivative of $\vec{r}$ wrt to v

$$S = \iint_D |r_u \times r_v| \, dA$$

$du\,dv$ or $dv\,du$

Derivative of $\vec{r}$ w/ respect to u

This is a place Holder
for Explaining why
we use this formula for calculation
of Area when we have a parametric
surface.
I will fill it later but you do NOT need
it for your exam (I explained it in the
class too)

Definition: If smooth parametric surface S is given by the eq.

$$r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \qquad (u,v) \in D$$

and S is just covered once as (u,v) ranges throughout the Parameter domain D, then the surface area of S is:

$$A(S) = \iint_D |r_u \times r_v| \, dA$$

where $r_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$

$$r_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

Example: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under plane $z = 9$.

Solution: $x(u,v) = u$, $y(u,v) = v$, $z = u^2 + v^2$

$$\Rightarrow r(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k} \Rightarrow \begin{cases} r_u(u,v) = i + 2u\vec{k} \\ r_v(u,v) = \vec{j} + 2v\vec{k} \end{cases}$$

$$\Rightarrow r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$$\Rightarrow |r_u \times r_v| = \sqrt{4u^2 + 4v^2 + 1}$$

$\Rightarrow A(S) = \iint_D \sqrt{4u^2+4v^2+1} \, dA \longrightarrow$ we can convert this to Polar Cords.

$z = u^2 + v^2$

$u = r\cos\theta$

$v = r\sin\theta$

$\Rightarrow z = r^2$

$\Rightarrow 9 = r^2$

$\Rightarrow r = +3$

$$\Rightarrow A(S) = \int_0^{2\pi}\int_0^3 \sqrt{1+4(r^2)} \, . r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2} \, dr\,d\theta = \frac{\pi}{6}(37\sqrt{37} - 1)$$

↳ After Solving

Example: find the surface area of a cone w/ the eq. $x^2+y^2=z^2$.

solution: we can parametrize $x^2+y^2=z^2$ using

$$\begin{cases} x(u,v) = u\cos v \\ y(u,v) = u\sin v \\ z(u,v) = v \end{cases} \Rightarrow r(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + v\,\vec{k}$$

$$\Rightarrow \left.\begin{aligned} r_u &= \cos v\,\vec{i} + \sin v\,\vec{j} + \vec{k} \\ r_v &= -u\sin v\,\vec{i} + u\cos v\,\vec{j} \end{aligned}\right\} \Rightarrow r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

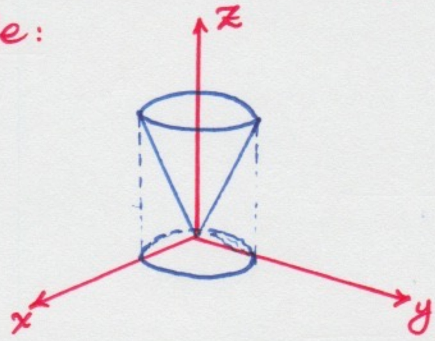
$$\Rightarrow r_u \times r_v = (u\cos v)\,i + (u\sin v)\,\vec{j} + u\,\vec{k}$$

$$\Rightarrow |r_u \times r_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{u^2+u^2}$$

$$= u\sqrt{2}$$

$$\Rightarrow A(S) = \int_0^{2\pi}\int_0^{r} u\sqrt{2}\,du\,dv = \pi\sqrt{2}\,r^2$$

Note: We use polar coordinates because the projection of the surface on the plane is a circle:



we pick the upper bound of the integral as r (for u) because $z=r$ and the area of the surface depends on where we pick r or (z).