Introduction:

The relationship b/w surface integrals and surface area is much the same as the relationship b/w line integrals and the arc length.

$$l = \int ds = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \cdot dt \qquad a \le t \le b$$

we also talked about line integrals along a curve, c:

$$\int_c f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)}\; dt$$

$$\text{when } \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

Then we took this idea further and we calculated the line integral in a vector field (tangential component of the vector field) which is really the work done to move the particle along the path c,

Also, in the previous section we learned how to calculate the area of a surface using $A(S) = \iint_S ds = \iint_D |r_u \times r_v|\, dA$

In this chapter, we want to learn dealing w/ surface integrals such as

$$\iint_S f(x,y,z)\, ds \quad \text{and} \quad \iint_S \vec{F} \cdot \vec{ds}$$

scalar function   vector field.

very similar to the idea of line integrals, if we go ahead and write the surface $f(x,y,z)$ in terms of its parametric equivalent, we will have

$$f(\vec{r}(u,v)) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

Now we can easily calculate $\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \cdot |r_u \times r_v| \cdot dA$

Example: Compute the surface integral $\iint_S x^2 \, ds$ where S is the unit sphere $x^2+y^2+z^2=1$.

Solution:

$\begin{cases} \phi = u \\ \theta = v \end{cases} \longrightarrow x(u,v) = \sin u \cos v \ , \ y(u,v) = \sin u \sin v \ , \ z(u,v) = \cos v$

$\Rightarrow r(u,v) = \sin u \cos v \, \vec{i} + \sin u \sin v \, \vec{j} + \cos v \, \vec{k} \qquad 0 \le u \le \pi, \ 0 \le v \le 2\pi$

$$\iint_S x^2 \, ds = \iint_D \sin^2 u \cos^2 v \cdot |r_u \times r_v| \, dA$$

$$|r_u \times r_v| = ? \qquad |r_u \times r_v| = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & 0 \\ -\sin u \sin v & \sin u \cos v & -\sin v \end{vmatrix} = \sin u$$

$$\Rightarrow \iint_D \sin^2 u \cos^2 v \cdot |r_u \times r_v| \, dA = \int_0^{\pi} \int_0^{2\pi} \sin^3 u \cos^2 v \, dv \, du = \frac{4\pi}{3}$$

Example: Evaluate $\iint_S y \, ds$ where S is the surface $z = x + y^2$, $0 \le x \le 1$ and $0 \le y \le 2$.

Solution: Since $z = x + y^2$ (one variable is written in terms of two other variables), we can pick the trivial parametrization.

$x(u,v) = u$

$y(u,v) = v$

$z(u,v) = u + v^2$

$\begin{cases} 0 \le u \le 1 \\ 0 \le v \le 2 \end{cases}$

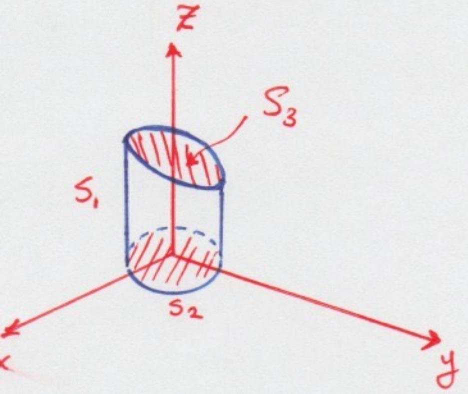
$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 2v \end{vmatrix} = i(-1) - (2v)\vec{j} + \vec{k}(1)$$

$$\Rightarrow |r_u \times r_v| = \sqrt{2 + 4v^2}$$

$$\Rightarrow \iint_S y \, ds = \int_0^1 \int_0^2 v\sqrt{2+4v^2} \, dv \, du \xrightarrow[\text{Ans}]{\text{Final}} \boxed{\frac{13\sqrt{2}}{3}}$$

Assume $2 + 4v^2 = t$

Ex. Evaluate $\iint_S z\,dS$ where S is the surface whose sides S_1 is given by $x^2+y^2=1$, whose bottom S_2 is given by the disk $x^2+y^2\leq 1$ in the plane $z=0$, and whose top S_3, is part of the plane $z=1+x$ that lies above S_2.



Let's start from the surface integral over $\underline{S_1}$:

$$\left.\begin{aligned} x(u,v) &= 1\cos u \\ y(u,v) &= 1\sin u \\ z(u,v) &= v \end{aligned}\right\} \quad \begin{aligned} &0\leq u\leq 2\pi \\ &0\leq v\leq 1+\cos u \end{aligned}$$

$$\Rightarrow r(u,v) = (\cos u)\vec{i} + (\sin u)\vec{j} + v\vec{k}$$

$$r_u\times r_v = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = i(\cos u) + j(\sin u) + k(0-0) \Rightarrow |r_u\times r_v| = 1$$

$$\Rightarrow \int_0^{2\pi}\int_0^{1+\cos u} v\,dv\,du = \int_0^{2\pi} \frac{v^2}{2}\Big|_0^{1+\cos u} du = \frac{3\pi}{2}$$

for surface $\underline{S_2}$, the bottom, the calculation is very easy:

$$z=0 \Rightarrow \iint_{S_2} z\,dS = 0$$

Now, we have to solve the given surface integral over $\underline{S_3}$, (The top):

top surface: $z=1+x$

$$\left.\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= 1+u \end{aligned}\right\} \Rightarrow r(u,v) = u\vec{i} + v\vec{j} + (1+u)\vec{k}$$

$$r_u\times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = i(0-1) - j(0-0) + k(1-0)$$

$$\Rightarrow |r_u\times r_v| = \sqrt{2}$$

So we know $|r_u \times r_v| = \sqrt{2}$

$$\iint_D \sqrt{2}\,(1+u)\,dA$$ $\longrightarrow$ we need to use polar coordinates as region D is circular.

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (1+r\cos\theta)\,r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{r^2}{2}+\frac{r^3}{3}\cos\theta \Big|_0^1 \,d\theta = \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{3}\cos\theta\right)d\theta = \pi\sqrt{2}$$

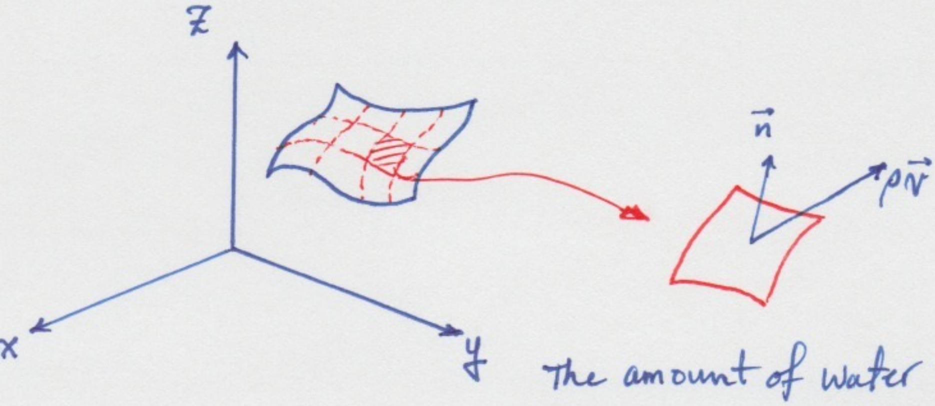
Now $$\iint_S z\,ds = \iint_{S_1} z\,ds + \iint_{S_2} z\,ds + \iint_{S_3} z\,ds = \boxed{\pi\sqrt{2} + 0 + \frac{3\pi}{2}}$$

Surface Integrals of vector Fields

In order to define surface Integrals of vector fields, we need to have oriented (Two sided) surface. (e.g. Möbius strip is not an oriented surface). However, almost all the surfaces that you will be dealing with will be oriented surfaces. for oriented surfaces, we have two choices for normal vector, $\vec{n}$.
The positive orientation is the vector that is coming outward from the surface.

Now, let's focus our attention on understanding and solving surface integrals of vector fields:

Suppose that we are given a surface made up of thin wires (A meshed surface like a fish net) and we are asked to Calculate the total mass of water that passes through the surface. The velocity of water at each point is given as $\vec{v}(x,y,z)$ and assume ρ, the density of water, is Constant.



The amount of water that passes through one of these little patches is equal to $\rho\vec{v}\cdot\vec{n}\,\underbrace{\Delta S}_{\text{Area of the little patch}}$

$\Rightarrow$ The approximation to total mass of water through the surface:

$$\sum \rho\vec{v}\cdot\vec{n}\,\Delta S$$

But this value is an approximation, if we want the exact value, then

$$= \iint_S \underbrace{\rho\vec{v}}_{\vec{F}}\cdot\vec{n}\,dS$$

$\Rightarrow$ Surface integral of a vector field $= \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$

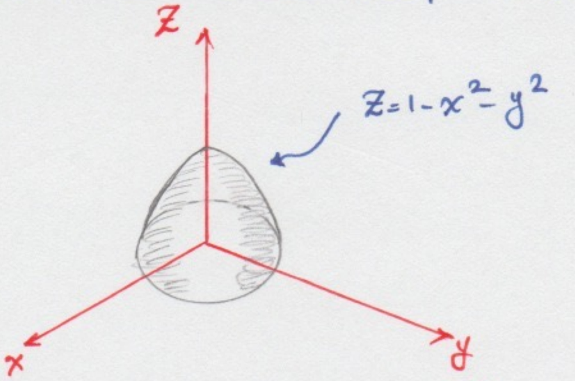
We still remember that $dS = |r_u \times r_v|\,dA$

$$\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$$

($\vec{r}(u,v)$ is the parametrization of surface S / position vector of surface S)

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(r_u \times r_v)\,dA$$

Example: Evaluate $\iint_S \vec{F}\cdot\vec{ds}$ where $F(x,y,z) = y\vec{i} + x\vec{j} + z\vec{k}$ and S is the boundary of paraboloid and the plane $z = 0$.



paraboloid: $z = 1 - x^2 - y^2$

$$\Rightarrow \vec{r}(u,v) = u\vec{i} + v\vec{j} + (1-u^2-v^2)\vec{k}$$

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = 1-u^2-v^2 \end{cases}$$

$$\vec{F}(x,y,z) = v\vec{i} + u\vec{j} + (1-u^2-v^2)\vec{k}$$

$$\Rightarrow r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = i(0+2u) - j(-2v-0) + k(1)$$

$$= 2u\vec{i} + 2v\vec{j} + \vec{k}$$

$$\iint_S \vec{F}\cdot\vec{ds} = \iint_D \vec{F}\cdot(r_u \times r_v)\,dA = \iint_D (v(2u) + u(2v) + 1 - u^2 - v^2)\,dA$$

$$= \iint_D (1 + 4uv - u^2 - v^2)\,dA \xrightarrow{\text{Polar}} = \int_0^{2\pi}\int_0^1 (1 + 4r^2\sin\theta\cos\theta - r^2)\,r\,dr\,d\theta$$

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$= \int_0^{2\pi}\int_0^1 r + \frac{4}{2}r^3\sin 2\theta - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} + \frac{2}{4}r^4\sin 2\theta - \frac{r^4}{4}\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{4} + \frac{1}{2}\sin 2\theta\right)d\theta = \boxed{\frac{\pi}{2}}$$

Do Not forget that this is only the surface integral of the top surface.

for bottom surface

$$\vec{n} = -\vec{k} \qquad \iint_S \vec{F}\cdot\vec{n}\,ds = \iint_S (y\vec{i} + x\vec{j} + z\vec{k})\cdot(-k)\,ds = \iint_{S,\ z=0} -z\,ds = 0$$

Final Ans: $0 + \frac{\pi}{2} = \frac{\pi}{2}$

Ex. Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2+y^2+z^2=1$.

Since the surface that we are dealing with is a sphere, we can use the following parametrization:

$$x(u,v) = \sin v \cos u \qquad 0 \le v \le \pi$$
$$y(u,v) = \sin v \sin u \qquad 0 \le u \le 2\pi$$
$$z(u,v) = \cos v$$

$$\Rightarrow r(u,v) = (\sin v \cos u)\vec{i} + (\sin v \sin u)\vec{j} + \cos v \vec{k}$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -\sin u \sin v & \sin v \cos u & 0 \\ \cos u \cos v & \cos v \sin u & -\sin v \end{vmatrix} = i(-\sin^2 v \cos u - 0) - j(\sin u \sin^2 v) + k(-\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v)$$

$$\underline{\underline{r_u \times r_v}} = -\left[i(\sin^2 v \cos u) + j(\sin^2 v \sin u) + (\sin v \cos v)\vec{k}\right]$$

$$\underline{\underline{\vec{F}}}(r(u,v)) = \cos v \vec{i} + \sin v \sin u \vec{j} + \sin v \cos u \vec{k}$$

$$\iint_D \vec{F}\cdot(r_u \times r_v) = -\int_0^{2\pi}\int_0^{\pi} (\sin^2 v \cos u \cos v + \sin^3 v \sin^2 u + \sin^2 v \cos u \cos v)\, dv\, du$$

$$= -\int_0^{2\pi}\int_0^{\pi} (2\sin^2 v \cos u \cos v + \sin^3 v \sin^2 u)\, dv\, du$$

$$= -2\int_0^{2\pi} \cos u\, du \int_0^{\pi} \sin^2 v \cos v\, dv - \int_0^{2\pi} \sin^2 u\, du \int_0^{\pi} \sin^3 v\, dv$$

After solving the above integrals: $\frac{4\pi}{3}$