An Add-on about Surface integrals:

last session we proved That the surface integrals may be calculated using:

$$\iint_S \vec{F}.\vec{n}.ds = \iint_D F.(r_u \times r_v)\,dA$$

your book uses one special case, when we have one variable in terms of the other two variables: $z = g(x,y)$

So, let's prove the formula that your book uses in these special cases so we know where it is coming from.

As we said previously, in these cases, we just use trivial Parametrization.

$x(u,v) = u$, $y(u,v) = v$, $z(u,v) = g(u,v)$

and we calculate $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

Since, we are dealing w/a trivial case, your book picks x and y instead of u, and v:

$x = x$, $y = y$, $z = g(x,y)$ $\Rightarrow \vec{r}(x,y) = x\vec{i} + y\vec{j} + g(x,y)\vec{k}$

Now, instead of $(r_u \times r_v)$ we should calculate $(r_x \times r_y)$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} \begin{array}{l} \\ \rightarrow \text{Components of } r_x \\ \rightarrow \text{Components of } r_y \end{array}$$

$$\Rightarrow r_x \times r_y = \left(0 - \frac{\partial g}{\partial x}\right)\vec{i} - \left(\frac{\partial g}{\partial y} - 0\right)\vec{j} + \vec{k}(1-0)$$

$$\Rightarrow (r_x \times r_y) = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}$$

Now, if we are interested in magnitude of $(r_x \times r_y)$ we can write it as: $|r_x \times r_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$

Also, if we have a vector field F, such that $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

cross product

$$\iint_D F \cdot (r_x \times r_y)\,dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Dot product

Stokes Theorem:

Stokes Theorem can be regarded as a higher dimension version of Green's Theorem.

in section 16.5, we prove That:

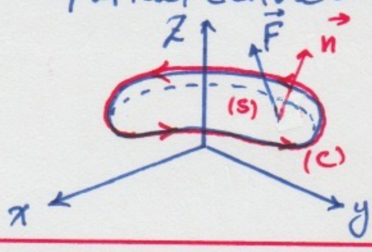
$$\oint_C F.dr = \iint_D \text{curl } F.\vec{k}\, dA$$ (one of the vector representations of Green's Theorem)

Now, instead of a plane, we go ahead and write the theorem for a surface integral in a vector field w/ closed boundary C.

Stokes Theorem:

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then:

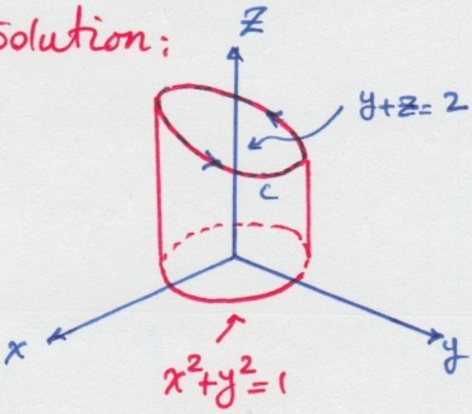
$$\int_C \vec{F}.\vec{dr} = \iint_S \text{curl } F.\vec{ds} = \iint_D \text{curl } \vec{F}.\vec{n}.ds$$



This Theorem tells us, if it is easier to calculate the value of the line integral compared to the surface integral or vice versa, we just go ahead and calculate the one that is easier. The results for both should be identical.

Example: Evaluate $\int_C F.dr$, where $F(x,y,z) = -y^2\vec{i} + x\vec{j} + z^2\vec{k}$ and C is The curve of intersection of plane $y+z=2$ and The cylinder $x^2+y^2=1$. (orient C to be Counterclockwise when viewed from above).

Solution:



$$F(x,y,z) = \underbrace{-y^2}_{P}\vec{i} + \underbrace{x}_{Q}\vec{j} + \underbrace{z^2}_{R}\vec{k}$$

These are P, Q, and R for $\vec{F}$, do NOT Confuse Them with P, Q, and R of $\nabla \times F$ which we will use later for This problem

Stokes: $\int_C F.dr = \iint_D \nabla \times F.(r_x \times r_y)\, dA$

$$\Rightarrow \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(1+2y) = (1+2y)\vec{k}$$

$$\Rightarrow \left.\begin{matrix} P=0 \\ Q=0 \\ R=1+2y \end{matrix}\right\} \text{For } \nabla \times F$$

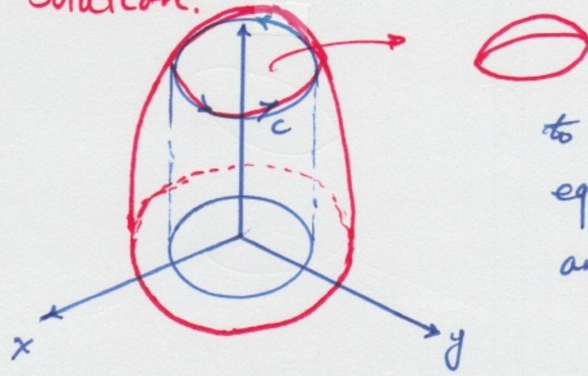
we also know $\left.\begin{matrix} y+z=2 \\ z=g(x,y) \end{matrix}\right\} \Rightarrow \begin{matrix} z=2-y \\ g(x,y)=2-y \end{matrix}$

$$\Rightarrow \iint_D \nabla \times F.(r_x \times r_y)\, dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA = \iint_D (1+2y)\, dA$$

Polar Coordinates for region D: $\int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r\, dr\, d\theta = \underline{\underline{\pi}}$

Example: Use Stokes Theorem to compute the integral $\iint_S \text{curl } F \cdot \vec{ds}$ where $F(x,y,z) = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$ and S is the part of sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

Solution:



to find the boundary of C, we just find the equation of the intersection points of $x^2+y^2=1$ and $x^2+y^2+z^2=4$:

$$\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow z^2=3 \Rightarrow z=\sqrt{3}$$

Now, we find the parametric eq. for the circle from intersection of cylinder & the sphere:

$x = r\cos t \quad (r=1) \Rightarrow x = \cos t \qquad 0 \le t \le 2\pi$

$y = r\sin t \quad (r=1) \Rightarrow y = \sin t \Rightarrow C: \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \sqrt{3}\,\vec{k}$

$z = \sqrt{3} \qquad \Rightarrow z = \sqrt{3} \qquad r'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$

Now, we have to parametrize F, the vector field as well:

$$F(\vec{r}(t)) = \underbrace{\sqrt{3}\cos t}_{xz}\,\vec{i} + \underbrace{\sqrt{3}\sin t}_{yz}\,\vec{j} + \underbrace{\sin t}_{xy}\cos t\,\vec{k}$$

$$\oint_C \vec{F}\cdot\vec{dr} = \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt = \int_0^{2\pi} (-\sqrt{3}\sin t\cos t + \sqrt{3}\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$