Introduction:

Previously, we wrote Green's Theorem in a vector version as:

$\int_C F\cdot n\,ds = \iint_D \operatorname{div} F(x,y)\,dA$, where C is a positively oriented boundary curve of the plane region D.

We can extend this Theorem to vector fields on $\mathbb{R}^3$ as

$\iint_S F\cdot n\,ds = \iiint \operatorname{div} F(x,y,z)\,dV$ where S is the boundary surface of the solid region E.

The Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given w/ positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F}\cdot\vec{ds} = \iiint_E \operatorname{div} F\cdot dV$$

Example: Find the flux of the vector field $F(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ over the unit sphere $x^2+y^2+z^2=1$.

Solution: The flux is given by $\iint_S \vec{F}\cdot\vec{ds} = \iint \vec{F}\cdot\vec{n}\,ds$ by divergence Theorem

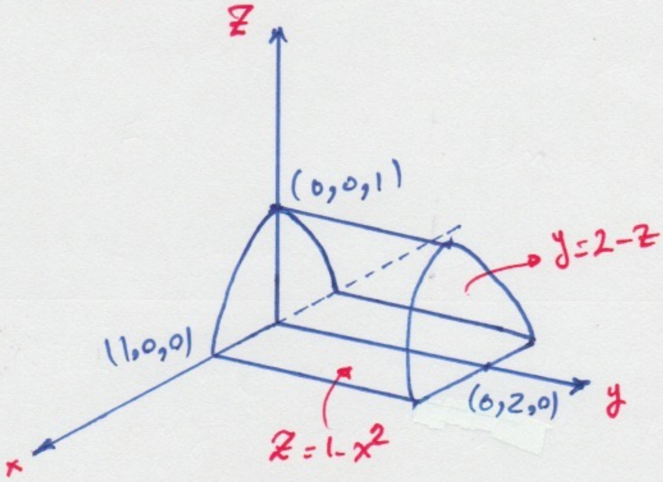
$\iint \vec{F}\cdot\vec{ds} = \iiint_E \operatorname{div} F\cdot dv$ $\quad \operatorname{div} F = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{ds} = \iiint_E 1\cdot dv = \text{Volume of the sphere (unit sphere)} = \left.\frac{4}{3}\pi r^3\right\}_{r=1}$$

$$\boxed{=\frac{4}{3}\pi}$$

Example: Evaluate $\iint_S F \cdot ds$ where $F(x,y,z) = xy\,i + (y^2 + e^{xz^2})\vec{j} + \sin(xy)\vec{k}$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

Solution:



it is very difficult to calculate the surface integral (we have to calculate the surface integral for all the sides (4 sides)). But, using the divergence Theorem we can easily evaluate $\iiint_E \operatorname{div} F \cdot dv$

$$\Rightarrow \operatorname{div} F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin xy)$$

$$= y + 2y + 0 = 3y$$

$$\Rightarrow \iint_S \vec{F} \cdot \vec{ds} = \iiint_E \operatorname{div} F \cdot dV = \iiint_E 3y \, dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \cdot dy\, dz\, dx$$

$$= 3 \int_{-1}^{1} \int_0^{1-x^2} \frac{y^2}{2} \Big]_0^{2-z} dz\, dx = 3 \int_{-1}^{1} \int_0^{1-x^2} \frac{(2-z)^2}{2} dz\, dx$$

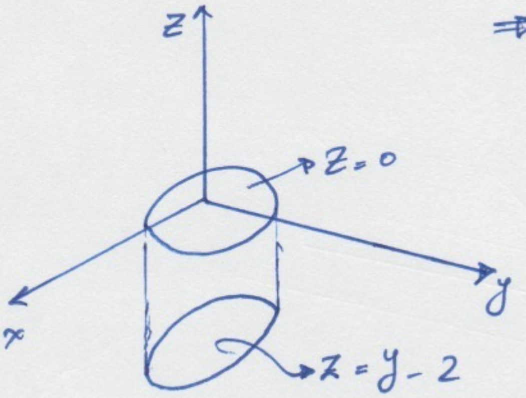
$$= \frac{3}{2} \int_{-1}^{1} \int_0^{1-x^2} \frac{(2-z)^2}{2} dz\, dx = \frac{3}{2} \int_{-1}^{1} \frac{-(2-z)^3}{3} \Big|_0^{1-x^2} dx$$

$$= -\frac{1}{2} \int_{-1}^{1} (2 - 1 + x^2)^3 - (2-0)^3 = -\frac{1}{2} \int_{-1}^{1} \left((x^2+1)^3 - 8\right) dx$$

$$= -\int_0^1 \left[(x^2+1)^3 - 8\right] dx = \cdots = \frac{184}{35}$$

Example: Use the divergence Theorem to calculate the surface integral $\iint_S \vec{F}\cdot\vec{ds}$, that is calculate the flux of F across S.

$F(x,y,z) = (xy + 2xz)\,\vec{i} + (x^2+y^2)\,j + (xy - z^2)\,k$ and S is the surface of the Solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=y-2$ and $z=0$.



$$\Rightarrow \quad \text{Div}\,F = \frac{\partial}{\partial x}(xy+2xz) + \frac{\partial}{\partial y}(x^2+y^2) + \frac{\partial}{\partial z}(xy-z^2)$$

$$\text{Div}\,F = (y+2z) + (2y) + (-2z)$$

$$\text{Div}\,F = 3y$$

$$\Rightarrow \iint_S F.ds = \iiint_E \text{Div}\,F.\,dv$$

$$\Rightarrow \iint_S \vec{F}.\vec{ds} = \int_0^{2\pi}\int_0^2\int_{y-2}^0 3y.r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{y-2}^0 3r^2\sin\theta\, dz\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2\int_{r\sin\theta-2}^0 r^2\sin\theta\, dz\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 r^2\sin\theta\, z\Big]_{r\sin\theta-2}^0 dr\,d\theta = 3\int_0^{2\pi}\int_0^2 0 - r^2\sin\theta(r\sin\theta-2)\,dr\,d\theta$$

$$= -3\int_0^{2\pi}\int_0^2 (r^3\sin^2\theta - 2r^2\sin\theta)\,dr\,d\theta = -3\int_0^{2\pi}\left(\frac{r^4}{4}\sin^2\theta - \frac{2}{3}r^3\sin\theta\right)\Big]_0^2 d\theta$$

$$= -3\int_0^{2\pi}\left(4\sin^2\theta - \frac{16}{3}\sin\theta\right)d\theta = \int_0^{2\pi}\left[16\sin\theta - 12\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

$$= \boxed{-12\pi}$$