A linear eq. in the variables $x_1, x_2, \dots, x_n$ is an equation that can be written in the following form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

where the coefficients $a_1, a_2, a_3, \dots a_n$ and the value of b are usually known (they can be real numbers or complex numbers).

$2x_1 - x_2 + 3x_3 = 5$ & $\sqrt{2}\,x_1 + \frac{\sqrt{6}}{\pi}x_2 = 4x_3$ are linear equations.

$2x_1x_2 - x_3 = 5$ & $\sqrt{x_1} + 2x_2 - x_3 = 6$ are not linear equations.

Two unknowns are being multiplied.

$\sqrt{x_1}$ is not a linear term.

A system of linear equations or a linear system is a collection of one or more linear equations involving the same variables

e.g. $\begin{cases} 2x_1 + x_2 = 4 \\ x_1 - x_2 = 2 \end{cases}$

A solution of the system is a list $(s_1, s_2, s_3, \dots, s_n)$ of numbers that makes each of the equations of the linear system a true statement if the values of $(s_1, s_2, \dots, s_n)$ are replaced w/ $(x_1, x_2, \dots, x_n)$ in the equations.

e.g. The solution of the previous system:

$$2x_1 + x_2 = 4$$
$$x_1 - x_2 = 2$$

is $(2,0)$, meaning $s_1 = 2$, $s_2 = 0$, should satisfy each eq. in the linear system.

$2(2) + (0) = 4$ ✓ True Statement

$(2) - (0) = 2$ ✓ True Statement

The set of all possible solutions is called the solution set. If two linear systems have the same solution set, they are called equivalent sets.

e.g. $\begin{cases} 2x_1 + x_2 = 4 \\ x_1 - x_2 = 2 \end{cases}$ & $\begin{cases} 4x_1 + 2x_2 = 8 \\ 2x_1 - 2x_2 = 4 \end{cases}$

are equivalent linear systems.

Graphical solution of a system of two linear equation:

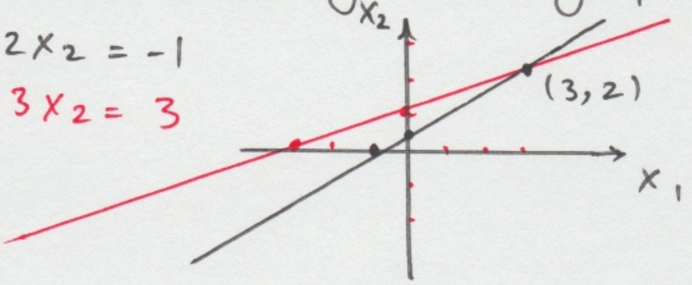
1/ when dealing with a linear system of two equations, we can consider each equation as an equation of a line.

If we graph both eq. in a coordinate system, the intersection point of the two lines is the solution of the system.

e.g. Solve the following system graphically.

$x_1 - 2x_2 = -1$

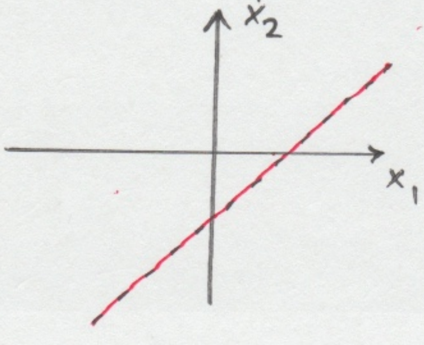
$-x_1 + 3x_2 = 3$



2/ when dealing with a linear system, the graphs of the equations might not necessarily intersect in just a single point. Sometimes the lines coincide, meaning there are infinitely many solutions to the equations:

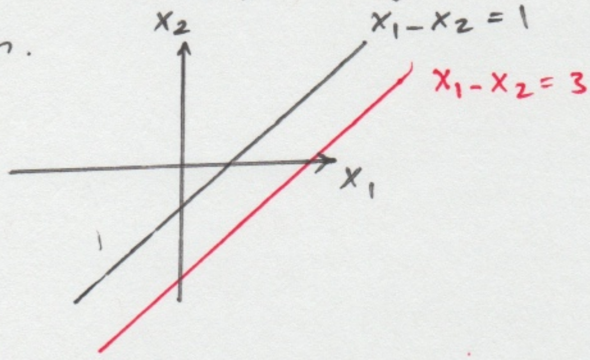
$x_1 - x_2 = 1$

$2x_1 - 2x_2 = 2$



3/ Additionaly, it is possible that the lines never intersect (when two lines are parallel). In that case, the system of equations does not have any solution.

e.g. $x_1 - x_2 = 1$
$x_1 - x_2 = 3$



Therefore, we may have three posibilities:

1/ A system of linear equation has no solution

2/ A system of linear equation has exactly one solution

3/ A system of linear equation has infinitely many solutions.

Matrix Notation:

All the necessary information of a linear system can be completely recorded in a rectangular array called a Matrix.

let's Think about the following system:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$$

The Coefficient Matrix or Matrix of coefficients can be written as:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

If we add the column on the right-hand side of the system of equations, we will have the augmented matrix of the system:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

The previously mentioned matrix has 3 rows & 4 columns. Therefore, The size of the matrix is 3x4 (The number of rows always come first & the number of columns always is next)

$$\begin{array}{cc}
\begin{array}{c}\text{w1}\\ \text{w2}\\ \text{ow3}\end{array} & \begin{bmatrix}1 & -2 & 1 & 0\\ 0 & 2 & -8 & 8\\ 5 & 0 & -5 & 10\end{bmatrix}\\
 & \begin{array}{cccc}\uparrow & \uparrow & \uparrow & \uparrow\\ \text{Column 1/} & \text{Column 2/} & \text{Column 3/} & \text{Column 4/}\end{array}
\end{array}$$

Solving linear Systems:

In This section, we would like to come up with a systemic procedure (algorithm) for solving a system of equations.

The strategy for solving a system is to replace a system with an equivalent system (one with the same solution set) that is easier to solve.

* There are three elementary row operations that may be applied on equations of a system w/o changing the solution set (having equivalent system)

1/ Replacement of a row by the sum of that row and a multiple of another row.

2/ Interchang (switch) two rows.

3/ Multiply all the elements in a row by a non-zero constant

The general idea to solve a system is:

To use x_1 term in the first equation to eliminate all the x_1 terms in other equations. Then to use x_2 term in the second equation to eliminate all the x_2 terms in subsequent equations. Then following the same procedure till you reach to x_n in the final equation and obtain a very simple equivalent system.

Let's see this procedure in action!

Ex. Solve the following system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad\quad - 5x_3 &= 10 \end{aligned} \xrightarrow[\text{Matrix}]{\text{Augmented}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Add (−5) times the first row to the third row (we do not need to worry about the second row in this case as the coefficient of x_1 in that equation is already 0). The short notation for this operation is $\boxed{-5R_1 + R_3}$ it may written down as:

$$\begin{array}{c} \\ \\ -5R_1+R_3 \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Add (−5) times row 2 to row 3 (to get rid of coefficient of x_2 in the Third equation

$$\begin{array}{c} \\ \\ -5R_2+R_3 \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now we have a triangular Matrix which corresponds to the following system:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned}$$

(This system is equivalent to the system that we started with) now the system has been simplified to:

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = -1$$

we can solve the system by back substitution:

$$x_3 = -1 \rightarrow x_2 - 4(-1) = 4 \rightarrow x_2 = 0$$
$$x_2 + 4 = 4$$
$$x_1 - 2(0) + (-1) = 0 \rightarrow x_1 = 1$$

or we can apply a similar procedure to avoid back substitution.

$$\begin{array}{c} R_3+R_1 \\ R_3+R_2 \\ \\ \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & -2 & 0 & +1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & +1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

which is equivalent to $\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$ Therefore, $(1, 0, -1)$ is the solution set.

if we plug back in the solution set $(1, 0, -1)$ we will see that it satisfies all the equations (the original equations)

Existence & Uniqueness

When we deal with a linear system, there are two very important & fundamental questions that we need to ask ourselves in order to understand the nature of a system.

question 1/ Is the system consistent, that is, does at least one solution exist?

question 2/ If a solution exists, is it the only one? that is, if a solution exists, is it unique?

Ex. Determine if the following system is consistent.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad\quad - 5x_3 &= 10 \end{aligned}$$

This is the system that we solved in the previous example.

we already know that $(1, 0, -1)$ is a solution of the system (the solution exists & the system is consistent.

In fact, we were able to uniquely find a value for x_3 and based on that we were able to determine x_2 uniquely and ultimately x_1 uniquely. Therefore, the system is unique.

Ex. Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \longrightarrow \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$-2R_1 + R_3$

switch row 1 & 2 because we do not want to have 0 in a pivot position

$$\xrightarrow{2R_2 + R_3} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

According to the third row (equation 3) $0 \cdot x_3 = 15$ or $0 = 15$

This is not possible

This means, there is a build-in contradiction in this system. This means, there is no value of x_1, x_2, x_3 (solution set) that can satisfy all these equations, so the system is inconsistent (not consistent)