In this section, we refine the method that we learned in the previous section to analyze any system of linear equations. First, we need to understand some definitions.

Non-zero row/Column: it is a row/column in a matrix that contains at least one nonzero entry.

Leading entry (in a row): refers to the leftmost nonzero entry, in a nonzero row.

Echelon form: A rectangular matrix is in echelon form if it has the following properties:

1/ All nonzero rows are above any rows of all zeros.

2/ Each leading entry of a row is in a column to the right of the leading entry of the row above it.

3/ All entry in a column below a leading entry are zeros (this is really a consequence of property 2)

e.g. $\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 3 & 5 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

reduced echelon form: is a matrix in echelon form that has the following two properties:

1/ the leading entry in each nonzero row is 1

2/ Each leading 1 is the only nonzero entry on its column.

e.g. $\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Important Note: Any nonzero matrix maybe row reduced (transformed by elementary row operations) into more than one matrix in <u>echelon form</u>.

However, the <u>reduced echelon form</u> is unique.

Theorem: uniqueness of the reduced echelon form:
Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot position:
since the reduced echelon form is unique, the leading entries are always in the same position in any echelon form obtained from a given matrix.

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains pivot position.

Ex. Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

we switch row 1 & 4 because we may not have zero in pivot position.

Add row 1 to row 2 and add 2x(row 1) to row 3

$-R_2 + R_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

switch row 3 and 4

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{3}{2}R_2 + R_3} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(we have multiplied row 2 by $(-3/2)$ and added to row 3.

In the previous example, we ended up w/ the following matrix:

Pivot positions (and indeed pivots)

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Columns

Now, this matrix is in its echelon form.

Notice that these numbers are NOT the same as the actual elements of A!

The row reduction algorithm:

e.g. Find the echelon form and then reduce the echelon form of the following matrix.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \xrightarrow[\text{row } 1 \& 3]{\text{switch}} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_2+R_3} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Now this matrix is in echelon form.

pivots (pivot positions)

* In order to find the reduced echelon form, we start with the rightmost pivot and working upward and to the left, to create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling factor.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[-6R_3+R_1]{-2R_3+R_2} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

All the elements above pivot position are zero & there is 1 in pivot position.

scale all the elements in row 2 to get 1 in pivot position.

$$9R_2+R_1 \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\text{scale} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(This matrix is the reduced echelon form of the original matrix)

Solutions of linear System:

Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of linear eq. associated w/ this reduced echelon matrix is:

$$\begin{aligned} x_1 \quad\quad -5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

* Variables x_1 & x_2 (which correspond to pivot columns) are called basic variables. The other variables (in this case x_3) are called free variables.

when a system of equations is consistent, as in the previous example, the solution set can be described explicitly by solving the reduced system of equations for basic variables in terms of free variables.

$x_1 = 1 + 5x_3$
$x_2 = 4 - x_3$
x_3 is free variable

this means whatever value we pick for x_3, we can plug it in the second eq. and find a value for x_2 & then in the first equation to find the value for x_1

This will be one solution (of many solutions) of the system.

Ex. Find the general solution of the linear system whose augmented matrix has been reduced to:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

This matrix is in echelon form but we need to find its reduced echelon form.

$$\begin{array}{c} R_3+R_2 \\ 2R_3+R_1 \\ \end{array}\begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow[\text{row 2}]{\text{scale}} \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$-2R_2+R_1 \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

this matrix is the reduced echelon form & corresponds to the system:

$$\begin{cases} x_1 + 6x_2 \qquad + 3x_4 = 0 \\ \qquad\qquad x_3 - 4x_4 = 5 \\ \qquad\qquad\qquad x_5 = 7 \end{cases}$$

The pivot columns are (column 1, 3, & 5), therefore, x_1, x_3, & x_5 are the basic variables & x_2 & x_4 are free variables.

$x_1 = -6x_2 - 3x_4$
x_2 is free
$x_3 = 5 + 4x_4$
x_4 is free
$x_5 = 7$

Existence & uniqueness question:

The echelon form of a matrix is just the right device for answering the two fundamental questions that we investigated in the previous section: 1/ existence of solution

2/ uniqueness of solution

Ex. Determine the existence and uniqueness of the solution of the following system:

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Ans. The augmented matrix for this system can be reduced to the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

pivot columns.

Basic variables: x_1, x_2, x_5

Free variables: x_3, x_4

* we do not see any eq. such as $0=1$ which would indicate an inconsistent system.

* Because of the free variables, we can be sure about the existence of solution. However, the solution is not unique as we can have different values for free variables.

Row reduction & echelon forms