Important properties of linear systems can be described w/ the concept & notation of vectors. let's begin with a very simple example.

$$\begin{cases} x_1 - x_2 = 2 \\ x_1 + x_2 = 4 \end{cases}$$

this system may be written as $x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$

A Matrix with only one column is called a column vector or just simply a vector.

$\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ & $\begin{bmatrix}2\\4\end{bmatrix}$ in the previous example are vectors in $\mathbb{R}^2$.

* vectors are equal if & only if their corresponding entries are equal. e.g. $\begin{bmatrix}2\\7\end{bmatrix} \neq \begin{bmatrix}7\\2\end{bmatrix}$

* Given two vectors such as $\vec{u}$ & $\vec{v}$, their sum is the vector $\vec{u}+\vec{v}$ obtained by adding the corresponding entries of $\vec{u}$ & $\vec{v}$:

$$\begin{bmatrix}3\\-1\end{bmatrix} + \begin{bmatrix}2\\ \frac{3}{2}\end{bmatrix} = \begin{bmatrix}2+3\\-1+\frac{3}{2}\end{bmatrix} = \begin{bmatrix}5\\ \frac{1}{2}\end{bmatrix}$$

* Given a vector $\vec{u}$ and a real number c, the scalar multiple of $\vec{u}$ by c is the vector $c\vec{u}$, obtained by multiplying each entry of $\vec{u}$ by c.

$$\underbrace{2}_{\text{Scalar}}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2(-1)\\2(3)\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$

Ex. Given $\vec{u} = \begin{bmatrix}1\\-2\end{bmatrix}$ & $\vec{v} = \begin{bmatrix}2\\-5\end{bmatrix}$, find $4\vec{u}$, $-3\vec{v}$ and $4\vec{u}-3\vec{v}$.

$$4\vec{u} = 4\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}4\\-8\end{bmatrix}$$

$$-3\vec{v} = (-3)\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}-6\\+15\end{bmatrix}$$

$$4\vec{u} - 3\vec{v} = 4\vec{u} + (-3)\vec{v} = \begin{bmatrix}4\\-8\end{bmatrix} + \begin{bmatrix}-6\\15\end{bmatrix} = \begin{bmatrix}-2\\+7\end{bmatrix}$$

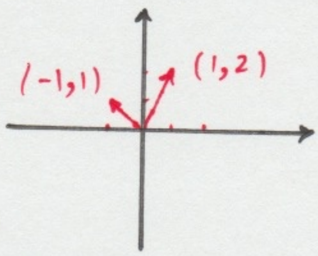
Geometric representation of vectors in $\mathbb{R}^2$ (rectangular coordinate system)

In $\mathbb{R}^2$, a vector may be represented by an arrow that starts from the origin of the coordinate system and ends at a point with coordinates the same as the components of the vector.

e.g $\begin{bmatrix} a \\ b \end{bmatrix}$ starts from $(0,0)$ and goes to the point (a,b)

e.g. Sketch the following vectors in $\mathbb{R}^2$.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$



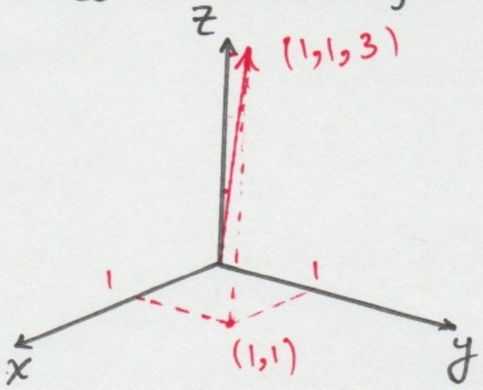
Geometric representation of vectors in $\mathbb{R}^3$:

vectors in $\mathbb{R}^3$ are column matrices with three entries.

e.g. $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

They can also be represented geometrically in 3D coordinate system by an arrow that start from the origin of the coordinate system $(0,0,0)$ and goes to the point (a,b,c).

e.g. vector $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ in a 3D coordinate system can be represented geometrically by



vectors in $\mathbb{R}^n$:

If n is a positive integer, $\mathbb{R}^n$ denotes the collection of all lists or n-tuples of n real numbers, usually written as $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$

properties of $\mathbb{R}^n$:

For all $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ & all scalars $\underline{c}$ and $\underline{d}$:

1/ $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2/ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3/ $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

4/ $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$ (where $-\vec{u}$ denotes $(-1)u$)

5/ $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

6/ $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

7/ $c(d\vec{u}) = (cd)\vec{u}$

8/ $1\vec{u} = \vec{u}$

Linear Combination

Given vectors $v_1, v_2, v_3, \ldots, v_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p$ the vector $\vec{w}$ defined by:

$\vec{w} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + \cdots + c_p v_p$ is a linear combination of $v_1, v_2, v_3, \ldots, v_p$ with weights $c_1, c_2, c_3, \ldots, c_p$ (any real numbe including zero)

e.g. let $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine if $\vec{b}$ can be written as a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

solution: let's introduce the weights x_1 & x_2

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = ?$$

in other word we need to ask if "There is a solution for This system of eqs.

$$\begin{cases} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3 \end{cases} \xrightarrow[\text{Matrix}]{\text{Augmented}} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

$$\begin{array}{r} \\ 2R_1 + R_2 \\ 5R_1 + R_3 \end{array} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ -R_2 + R_3 \end{array} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

the solution of This system is $\boxed{x_2 = 2}$, $x_1 + 2x_2 = 7$
$\Rightarrow x_1 + 4 = 7 \Rightarrow \boxed{x_1 = 3}$

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$ so $\vec{b}$ is a linear combination of $\vec{a}_1$ & $\vec{a}_2$

* what you just saw in previous example, may be written down as:

A vector eq. $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots + x_n\vec{a}_n = b$ has The same solution set as The linear system that its augmented matrix is $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$

Important Note: In particular, $\vec{b}$ can be generated by a linear combination of $a_1, \ldots, a_n$ if and only if there exists a solution to the linear system corresponding to the previous matrix.

Vector Equations

The Important Question:

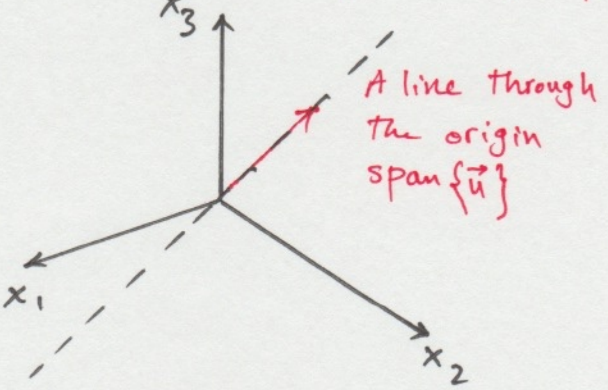
Now the question is, what is the set of all the vectors that can be generated or written as the linear combination of vectors in the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p\}$

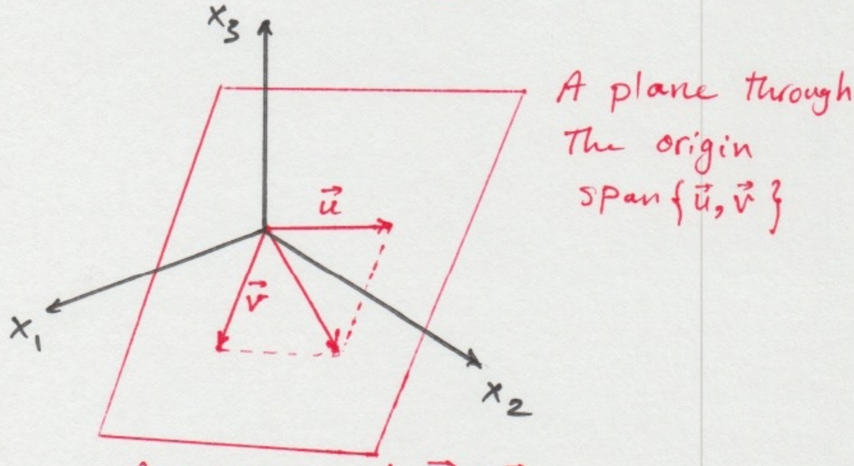
The set of all linear combinations of $v_1, \dots, v_p$ is denoted by $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$. Pay attention that we have p vectors but the size of each vector is n.

$\text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$ = The collection of all vectors that can be written down in the form of $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$ with $c_1, c_2, \dots, c_p$ scalars.

** Therefore, asking whether a vector $\vec{b}$ is in $\text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is the same as asking if the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{b}$ has a solution

Geometric description of $\text{span}\{\vec{u}\}$ & $\text{span}\{\vec{u}, \vec{v}\}$ in $\mathbb{R}^3$:





Assuming that $\vec{u}$ & $\vec{v}$ are non-zero and $\vec{v}$ is not a multiple of $\vec{u}$

Example: If $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ & $a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, then the span$\{\vec{a}_1, \vec{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$ in that plane.

Solution: In reality, the question is, "Does the eq. $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ have a solution?"

$$\begin{array}{r} \\ 2R_1+R_2 \\ -3R_1+R_3 \end{array}\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \xrightarrow{-6R_2+R_3} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

from the equation coming from the third row, $0=-2$ so the system does not have any solution.