The Matrix Equation $A\vec{x} = \vec{b}$

Section 1.4

Def. If A is an $m \times n$ matrix, with columns $a_1, \dots, a_n$ & if $\vec{x}$ is in $\mathbb{R}^n$, then the product of A & $\vec{x}$ denoted by $A\vec{x}$, is the linear combination of the columns of A using the corresponding entries in x as weights.

$$A\vec{x} = \underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \dots + x_n\vec{a}_n$$

example a) $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$$
$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

example b) $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

Note: This operation is defined only if the number of columns of A equals the number of entries (rows) in $\vec{x}$.

Example: For $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\vec{v}_1 - 5\vec{v}_2 + 7\vec{v}_3$ as a matrix times a vector.

Ans: $$\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{m \times 3 \text{ Matrix}} \underbrace{\begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}}_{3 \times 1 \text{ vector}}$$

The Matrix Equation $A\vec{x} = \vec{b}$

Ex. The following system of linear equations, the vector equation & the matrix equation are equivalent:

$$\begin{cases} x_1 + 2x_2 - x_3 = 4 \\ \quad -5x_2 + 3x_3 = 1 \end{cases}$$ system of linear equations

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ vector equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ Matrix equation

↑
This is called Matrix of coefficients.

This means the solution of the matrix eq. $A\vec{x} = \vec{b}$ is the same as the solution of the vector equation:

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{b}$$

and that has the same solution set as the system of linear equations whose augmented matrix is $[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n \ \vec{b}]$

Existence of Solution:

The definition of $A\vec{x}$ lead to the following useful facts

* The eq. $Ax = \vec{b}$ has a solution if & only if $\vec{b}$ is a linear combination of the columns of A.

The Matrix Equation $A\vec{x} = \vec{b}$

Ex. let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the eq. $A\vec{x} = \vec{b}$ consistent for all possible b_1, b_2, and b_3?

$$\begin{array}{c} \\ 4R_1+R_2 \\ 3R_1+R_3 \end{array} \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \longrightarrow \begin{array}{c} \\ \\ -\tfrac{1}{2}R_2+R_3 \end{array} \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 7 & 5 & 3b_1+b_3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1+b_2 \\ 0 & 0 & 0 & 3b_1+b_3-\frac{1}{2}(4b_1+b_2) \end{bmatrix}$$

This means $A\vec{x} = \vec{b}$ is not consistent for all values of $\vec{b}$

if $3b_1+b_3-\frac{1}{2}(4b_1+b_2) \neq 0$, The eq. is NOT consistent

This eq. is only consistent when $b_1 - \frac{1}{2}b_2 + b_3 = 0$

* If Matrix A had a pivot in all three rows, we would not care about the calculation in the augmented column because an echelon form of the augmented matrix could not have a row such that $[0 \; 0 \; 0 \; 1]$ (inconsistent)

Theorem: let A be an $m \times n$ matrix. Then, the following statements are logically equivalent (meaning either they are all true or all false)

* For each $\vec{b}$ in $\mathbb{R}^m$, the eq. $A\vec{x} = \vec{b}$ has a solution.
* Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A
* The columns of A span $\mathbb{R}^m$ (Every $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A span $\mathbb{R}^m$)

*** A has a pivot position in every row.

(Note: This theorem is about A, the coefficient matrix, not $[A, b]$, the augmented matrix.

↳ in reality, if $[A \;\; b]$ has pivot position in every row, then the eq. $A\vec{x} = \vec{b}$ may or may not be consistent.

Ex. (A more efficient way to calculate $A\vec{x}$)

Compute $A\vec{x}$ where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

we previously used the def. as

$$A\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} x_3 = \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

However, using the concept of dot product gives us a much simpler way of calculating $A\vec{x}$

The dot product of two vectors is the same as the sum of products of corresponding entries.

In this case, the first entry in the product $A\vec{x}$ is the dot product of the First Row of A by the entries in $\vec{x}$ (column)

$$\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \\ \end{bmatrix}$$

The second entry of $A\vec{x}$ is the dot product of the second row of A by the entries in $\vec{x}$ (column).

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

and ...

e.g. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 + 6 - 7 \\ 0 - 15 + 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 - 21 \\ 32 + 0 \\ -20 + 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

↳ by def. This is an identity matrix (a matrix with elements on the diagonal being 1, and all other elements 0). Identity Matrix is shown with $\underset{\sim}{I}$.

Properties of Matrix-vector product $A\vec{x}$:

If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ (meaning they have n entries), and c is a scalar, then:

a. $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

b. $A(c\vec{u}) = c(A\vec{u})$

"The proof is provided for you in section 1.4, page 39 of your book"