Homogeneous Linear Systems:

A system of linear equations is said to be homogeneous if it can be written in the form $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix & $\vec{0}$ is the zero vector in $\mathbb{R}^m$. Such a system always has at least one solution $\vec{x} = \vec{0}$ (The zero vector in $\mathbb{R}^n$)

This solution is called the Trivial Solution.

Now, the important question is whether there exists a nontrivial solution (A non-zero vector that satisfies $A\vec{x} = \vec{0}$)

From Section 1.2 we remember:

The homogeneous eq. $A\vec{x} = \vec{0}$ has a nontrival solution if & only if the equation has at least one free variable.

Ex. Determine if the following homogeneous system has a non-trivial solution. (Describe the solution set)

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases} \quad \begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-5/3 R_2 + R_3} \begin{bmatrix} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \\ x_3 \text{ is free} \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 \vec{v} \longrightarrow$$

This shows that every solution of this system is a scalar multiple of vector $\vec{v}$. The trivial solution is obtained by choosing $x_3 = 0$

* Geometrically, the solution set is a line through $\vec{0}$ in $\mathbb{R}^3$.

Ex. Describe all solutions of the homogeneous system $10x_1 - 3x_2 - 2x_3 = 0$

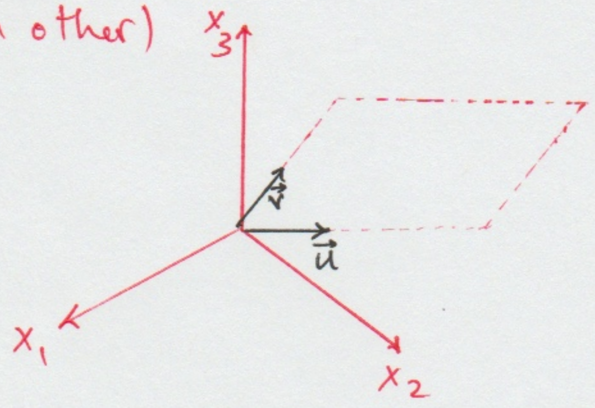
solution: A single linear equation can be treated as a very simple system of equations. There is no need for Matrix notation.

$x_1 = 0.3x_2 + 0.2x_3$ (x_2 & x_3 are free variables)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

So every solution is a linear combination of vectors $\vec{u}$ & $\vec{v}$ (the solution is a plane since $\vec{u}$ & $\vec{v}$ are not multiple scalar of each other)



this means, the solution of a homogeneous eq. $A\vec{x} = \vec{0}$ can always be expressed explicitly as span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ for suitable vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$

Solutions of nonhomogeneous systems ($A\vec{x}=\vec{b}$)

In general, for nonhomogeneous linear systems, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

Ex. Describe all solutions of $A\vec{x}=b$, where $A=\begin{bmatrix}3&5&-4\\-3&-2&4\\6&1&-8\end{bmatrix}$ & $\vec{b}=\begin{bmatrix}7\\-1\\-4\end{bmatrix}$.

$$\begin{bmatrix}3&5&-4&7\\-3&-2&4&-1\\6&1&-8&-4\end{bmatrix}\longrightarrow\begin{bmatrix}3&5&-4&7\\0&3&0&6\\0&-9&0&-18\end{bmatrix}\longrightarrow\begin{bmatrix}3&5&-4&7\\0&1&0&2\\0&-1&0&-2\end{bmatrix}\longrightarrow$$

$$\begin{bmatrix}3&5&-4&7\\0&1&0&2\\0&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}3&0&-4&-3\\0&1&0&2\\0&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&-4/3&-1\\0&1&0&2\\0&0&0&0\end{bmatrix}$$

$$x_1-\frac{4}{3}x_3=-1$$

$$x_2=2$$

x_3 is free

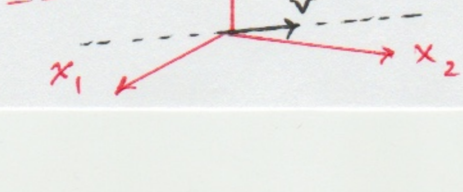
$$x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1+\frac{4}{3}x_3\\2\\x_3\end{bmatrix}=\underbrace{\begin{bmatrix}-1\\2\\0\end{bmatrix}}_{\vec{p}}+\underbrace{\begin{bmatrix}4/3\\0\\1\end{bmatrix}}_{\vec{v}}x_3$$

The solution for this system Then, is $\vec{p}+t\vec{v}$

scalar

which is also a line that is shifted by vector $\vec{p}$.

(recall $t\vec{v}$ is just a line along vector $\vec{v}$ that passes through the origin)



the solution is $t\vec{v}$ but shifted with $\vec{p}$ (still a line)

Theorem: If $A\vec{x} = \vec{b}$ has a solution, then the solution is obtained by translating (shifting) the solution set of $A\vec{x}$ using any particular solution $\vec{p}$ of $A\vec{x} = \vec{b}$ for the translation.

(This theorem only applies if there exists at least one non-zero solution for $A\vec{x} = \vec{b}$.

To write down a solution set (of a consistent system) in parametric vector form, we follow this algorithm:

1/ Row reduce the augmented matrix to reduced echelon form.

2/ Express basic variables in terms of free variables.

3/ Write a typical solution $\vec{x}$ as a vector whose entries depend on the free variables, if any.

4/ Decompose x into a linear combination of vectors (with numeric entries) using free variables as parameters.