In this section, we want to study the homogeneous equations from a different perspective by writing them as vector equations.

Def.: The set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = 0$ has only trivial solution (no choice of $x_1, x_2, \ldots, x_p$ leads to a non-trivial solution)

The set $\{v_1, v_2, \ldots, v_p\}$ is said to be linearly dependent if there exist at least one set of weights $c_1, c_2, \ldots, c_p$ (not all zero) such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$

Ex. let $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$ and $v_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$. Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent. If possible, find a linear dependence relation among v_1, v_2, and v_3.

We need to find a non-trivial solution for the eq.

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} + x_3\begin{bmatrix}2\\1\\0\end{bmatrix} = 0 \qquad \begin{bmatrix}1 & 4 & 2 & 0\\2 & 5 & 1 & 0\\3 & 6 & 0 & 0\end{bmatrix} \longrightarrow$$

$$\begin{bmatrix}1 & 4 & 2 & 0\\0 & -3 & -3 & 0\\0 & -6 & -6 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 4 & 2 & 0\\0 & -3 & -3 & 0\\0 & 0 & 0 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 4 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 4x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

x_3 is a free variable

Any solution in the form of $x_1 = 2x_3$, $x_2 = -x_3$ (for $x_3 \neq 0$) will lead to a non-trivial soln.

e.g. $x_3 = 5$, $x_2 = -5$, $x_1 = 10$ is a solution. This means the vectors are linearly dependent.

set of one or two vectors:

* A set containing only one vector, $\vec{v}$, is linearly independent if & only if $\vec{v}$ is not a zero vector.
(because the eq. $x_1\vec{v} = \vec{0}$ only has one trivial solution when $\vec{v} \neq 0$)

* A set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of other (& linearly independent if & only if neither of vectors is a multiple of the other one)

Ex. Determine if the following sets of vectors are linearly independent.

a) $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \rightarrow$ linearly dependent

b) $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \rightarrow$ linearly independent

Geometrically, two vectors are linearly dependent if & only if they lie on the same line through the origin.

* The previous example about the linear dependence/independence may be extended to the following theorem.

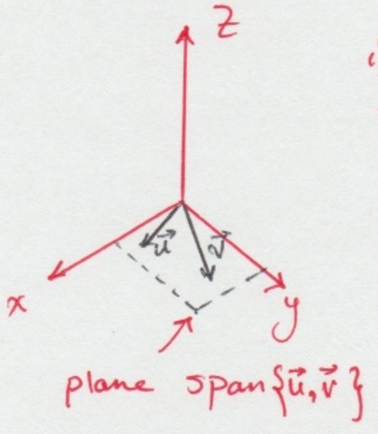
Theorem: A set $\{v_1, v_2, \ldots, v_p\}$ (of two or more vectors) is linearly dependent if & only if at least one of the vectors in the set is a linear combination of the others (This does NOT mean that every vector in a linearly dependent set is a linear combination of the other vectors).

e.g. In the previous example $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, & $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightarrow v_3$ is not a linear combination of either v_1, or v_2 but it is a linear combination of v_1 & v_2.

$-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, Therefore, $(-2, 1, -1)$ is a solution of $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$

Ex. let $\vec{u}=\begin{bmatrix}3\\1\\0\end{bmatrix}$ & $\vec{v}=\begin{bmatrix}1\\6\\0\end{bmatrix}$. Describe the set spanned by $\vec{u}$ & $\vec{v}$ and explain why a vector like $\vec{w}$ is in span$\{\vec{u},\vec{v}\}$ if & only if $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent.

$\vec{u}$ and $\vec{v}$ are linearly independent (They are not multiple of each other). Therefore, The span of $\vec{u}$ & $\vec{v}$ $\rightarrow$ Span$\{\vec{u},\vec{v}\}$ is a plane in $\mathbb{R}^3$ (since the third components of $\vec{u}$ & $\vec{v}$ are zero) The plane of Span$\{\vec{u},\vec{v}\}$ is in x_1x_2 plane.



if $\vec{w}$ is a linear combination of $\vec{u}$ & $\vec{v}$, then by the previous Theorem $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent.

Conversly, let's assume $\{\vec{u},\vec{v},\vec{w}\}$ is linearly dependent. The question is that wether $\vec{w}$ is in span$\{\vec{u},\vec{v}\}$ or Not?

If $\{u,v,w\}$ is linearly dependent, there must be a vector in $\{\vec{u},\vec{v},\vec{w}\}$ that is a linear combination of proceeding vectors. $\vec{u}\neq\vec{0}$, and $\vec{v}$ is not a multiple of $\vec{u}$. Therefore, $\vec{w}$ is that vector (linear combination of $\vec{u}$ & $\vec{v}$) and it must be in Span$\{\vec{u},\vec{v}\}$

* The following two theorems describe the situation in which the linear dependence of a set is automatic.

* **Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent (That is, any set $\{v_1, v_2, \dots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$)

<u>Proof</u>: If we construct matrix A as $A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p]$, then A is $n \times p$ and the equation $A\vec{x} = \vec{0}$ corresponds to a system of n eqs. and p unknowns. If $p > n$, then there are more variables than equations. Therefore, the system must have free variables. Hence, $A\vec{x} = \vec{0}$ has non-trivial solution and the columns are going to be linearly dependent.

* **Theorem:** If a set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

<u>Proof</u>: If a vector in the set like v_j is zero $\vec{v}_j = \vec{0}$, then we can write

$$0\vec{v}_1 + 0\vec{v}_2 + \dots + 1\vec{v}_j + \dots + 0\vec{v}_p = \vec{0}$$

Therefore, there exists at least a non-zero solution ($\vec{x}$ such that not all the elements of it are zero) such that:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_j\vec{v}_j + \dots + x_p\vec{v}_p = 0$$

which means the vectors are linearly dependent.

Ex. The vectors $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}$ are linearly dependent, why?

Since we have three vectors and two entries (2 eqs. & 3 unknowns)

Think about $x_1\begin{bmatrix}2\\1\end{bmatrix} + x_2\begin{bmatrix}4\\-1\end{bmatrix} + x_3\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

$\rightarrow$ 2 eqs., 3 unknowns
This means we must have free variables, therefore, the vectors are linearly dependent.

Ex. Determine by inspection if the given set is linearly dependent.

a) $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ b) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$ c) $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

a. linearly dependent (3 eqs (entries) and 4 unknowns (variables))

b. linearly dependent (one vector is zero)

c. linearly independent