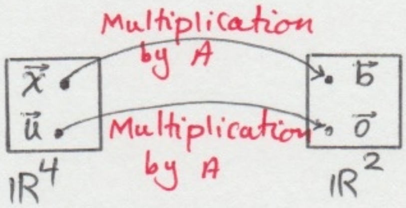
Think about the following Matrix by vector multiplication:

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{\text{Matrix } A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{vector } \vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{\text{Vector } \vec{b}} \quad \& \quad \underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{\text{Matrix } A} \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}}_{\text{Vector } \vec{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0} \text{ vector}}$$

We can say that multiplication by A $\boxed{\text{transforms}}$ vector $\vec{x}$ into $\vec{b}$ and transforms vector $\vec{u}$ into the zero vector.



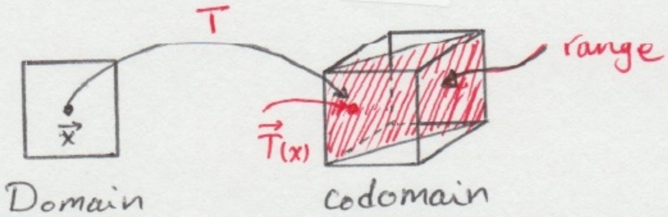
You can think of This as a function $x \xrightarrow{f} f(x)$

The Correspondence from $\vec{x}$ to $A\vec{x}$ is a function from one set of Vectors (in $\mathbb{R}^4$) to another set (in $\mathbb{R}^2$)

A Transformation (or a function or mapping) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule That assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called codomain of T.

The notation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ indicates That The domain of T is $\mathbb{R}^n$ & The codomain is $\mathbb{R}^m$. For x in $\mathbb{R}^n$, the vector $T(\vec{x})$ in $\mathbb{R}^m$ is called The image of $\vec{x}$ (under the action of T).

The set of all images of $T(x)$ is called the range of T.



This means for each $\vec{x}$ in $\mathbb{R}^n$, $T(\vec{x})$ is calculated as $A\vec{x}$, where A is an $m\times n$ matrix. For simplicity, we sometimes show the matrix transformation by $\vec{x} \longmapsto A\vec{x}$.

domain of T is $\mathbb{R}^n$ (A has n columns)

Codomain of T is $\mathbb{R}^m$ (A has m rows)

* The range of T is the set of all linear combinations of the columns of A, because each image $T(\vec{x})$ is in the form of $A\vec{x}$.

Example: let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ so that

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a) Find $T(\vec{u})$, that is the image of $\vec{u}$ under transformation T.

b) Find $\vec{x}$ in $\mathbb{R}^2$ whose image under T is $\vec{b}$.

c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$? (uniqueness problem)

d) Determine if c is in the range of transformation T. (existence problem)

Ans. a) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

b) we need to solve $A\vec{x} = \vec{b} \longrightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

we can row reduce the augmented matrix

$$\begin{array}{c} \\ -3R_1+R_2 \\ R_1+R_3 \end{array} \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \longrightarrow \begin{array}{c} \\ \\ -R_2+R_3 \end{array} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \\ +3R_2+R_1 \\ \end{array} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $x_1 = \frac{3}{2}$ & $x_2 = -\frac{1}{2}$. This means the image of this vector $\vec{x}$, $\vec{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ under T is vector $\vec{b}$.

c) The above eq. has a unique solution so there is only $\vec{x}$ (we found it!) whose image is $\vec{b}$.

d) This means, is there a vector like $\vec{x}$ such that $A\vec{x} = \vec{c}$. Again, we row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

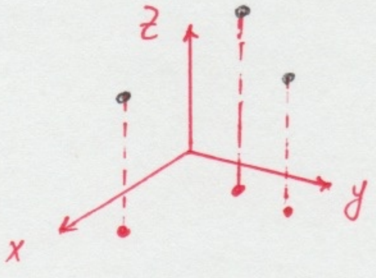
we can stop here!

eq. 3 from this matrix is $0x_1 + 0x_2 = 5$. this is not possible! The system is inconsistent & $\vec{c}$ is not in the range of T.

Ex. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Then interprete the transformation $\vec{x} \longrightarrow A\vec{x}$ geometrically.

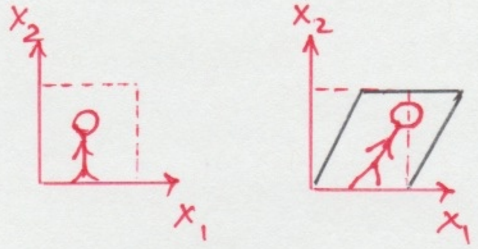
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

This means that this transformation, projects points in $\mathbb{R}^3$ onto the Two dimensional x_1x_2 plane.



Ex. let $A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$. Interprete the transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ geometrically.

$$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1+3x_2\\ x_2\end{bmatrix}$$ $\longrightarrow$ This transformation is called shear Transformation. This transformation leaves the second coordinate of each point intact but increases the first coordinate of the point causing it to shift to the right.



Linear Transformations

A Transformation or mapping T is linear if:

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$, and $\vec{v}$ in the domain of T
2) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c, and all $\vec{u}$ in the domain of T.

* Remember: every matrix transformation is a linear transformation.
* These properties lead to the following facts:

$$T(\vec{0}) = \vec{0}$$
$$T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$

for all vectors $\vec{u}$, and $\vec{v}$ in the domain of T and all scalars $\underline{c}$, and $\underline{d}$.

* repeated application of these properties produce a generalization:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_pT(\vec{v}_p)$$

This is referred to as superposition principle.

Example: Given a scalar r, define $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$.
T is called a **contraction** when $0 \le r \le 1$ and a **dialation** when $r > 1$.
Let $r = 3$, and show that T is a linear transformation.

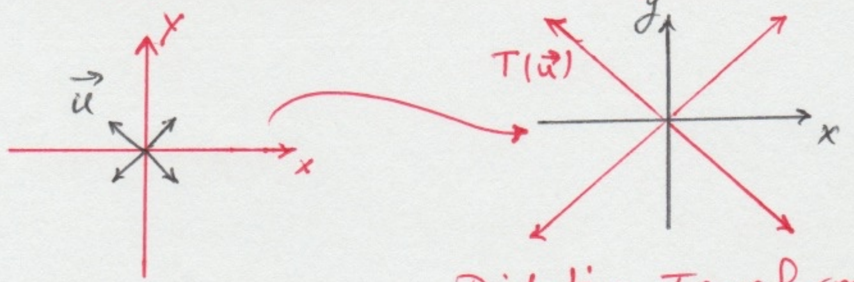
Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^2$ and $\underline{c}$ and $\underline{d}$ be scalars.

Then $T(c\vec{u} + d\vec{v}) = 3(c\vec{u} + d\vec{v})$ (by definition of T)

$= 3(c\vec{u}) + 3(d\vec{v}) = c(3\vec{u}) + d(3\vec{v})$

$= cT(\vec{u}) + dT(\vec{v})$ (by the vector arithmatics)

Therefore, T is a linear transformation because it satisfies the properties discussed about linear transformations.



Dialation Transformation

Ex. linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under T of $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\vec{u} + \vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

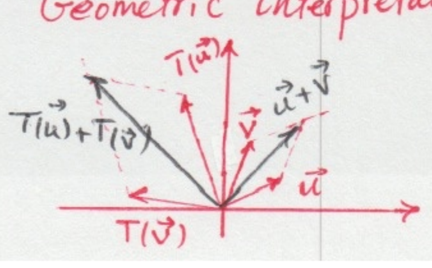
$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $T(\vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$T(\vec{u} + \vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ +6 \end{bmatrix}$

As we see $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

superposition principle

Geometric Interpretation:



The linear transformations do not have to be necessarily geometrical. here is an example of a linear transformation that converts one type of data to another.

ex. A company manufactures two products, B, & C. using data below, we can construct a unit cost matrix.

$$\vec{u} = [\vec{b} \;\; \vec{c}] \longrightarrow \vec{U} = \begin{array}{cc} \begin{bmatrix} 0.45 & 0.4 \\ 0.25 & 0.3 \\ 0.15 & 0.15 \end{bmatrix} & \begin{array}{l} \rightarrow \text{Materials} \\ \rightarrow \text{Labor} \\ \rightarrow \text{overhead} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{prod. B} & \text{prod. C} \end{array} & \end{array}$$

let $\vec{x} = (x, x_2)$ be a production vector corresponding to x_1 dollars of product B and x_2 dollars of product C, T is defined as $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by

$$T(\vec{x}) = \vec{U}\vec{x} = x_1 \begin{bmatrix} 0.45 \\ 0.25 \\ 0.15 \end{bmatrix} + x_2 \begin{bmatrix} 0.4 \\ 0.3 \\ 0.15 \end{bmatrix} = \begin{bmatrix} \text{Total cost of Material} \\ \text{Total cost of labor} \\ \text{Total cost of overhead} \end{bmatrix}$$

T transforms a list of production quantities (measured in dollars) into a list of total costs. pay attention to the following:

1/ if the production is increased by a factor, (say a factor of 4), from x to $4x$, then the cost will increase by the same factor from $T(\vec{x})$ to $4T(\vec{x})$.

2/ If x and y are production vectors, then the total cost vector that is associated w/ the combined production $x+y$ is precisely the sum of the cost vectors $T(\vec{x})$ and $T(\vec{y})$.