The Matrix of a Linear Transformation

where a linear transformation arises geometrically, we always want to find an $m \times n$ matrix A that the matrix transformation $\vec{x} \longrightarrow A\vec{x}$ describes.

In order to find matrix A, it is sufficient to know what the transformation T does to the columns of $n \times n$ identity matrix I_n.

Ex. The linear transformation T, transforms the column of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ from $\mathbb{R}^2$ into $\mathbb{R}^3$ as followed:

$T(\vec{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ & $T(\vec{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$, find Matrix A.

Solution: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2$

since T is a linear transformation: $T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$

$$= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0x_2 \end{bmatrix}, \text{ Therefore, } A = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}$$

Basically, we can put the vectors $T(\vec{e}_1)$, $T(\vec{e}_2)$ into the columns of matrix A and write:

$$T(\vec{x}) = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

Theorem: let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_2) \end{bmatrix}$$

Proof: we can write $\vec{x} = I_n\vec{x} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$

$$= \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}$$

* standard matrix for linear transformation

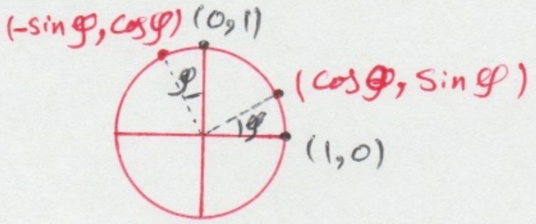
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = x_1 \begin{bmatrix} a \\ c \end{bmatrix} + x_2 \begin{bmatrix} b \\ d \end{bmatrix}$$

Ex. Find The standard matrix A for the dilation transformation $T(\vec{x}) = 3\vec{x}$ for $\vec{x}$ in $\mathbb{R}^2$.

Soln: $T(\vec{e}_1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation That rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counter-clockwise rotation for a positive angle. Find The standard matrix A of This transformation.

If we rotate the points on $\overset{\vec{e}_1}{(1,0)}$ & $\overset{\vec{e}_2}{(0,1)}$ position, we will have



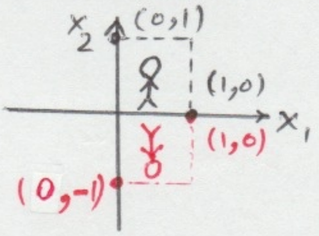
Therefore, The matrix of rotation would be $\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

Geometric Linear Transformations of $\mathbb{R}^2$:

usually linear transformations in $\mathbb{R}^2$ may be described geometrically. Since The transformations are linear, They may be determined completely by what they do to The columns of I_2 ($\vec{e}_1$ & $\vec{e}_2$).

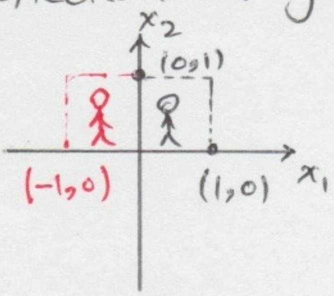
The following table shows what a transformation does to The unit square:

1- Reflection through x_1-axis



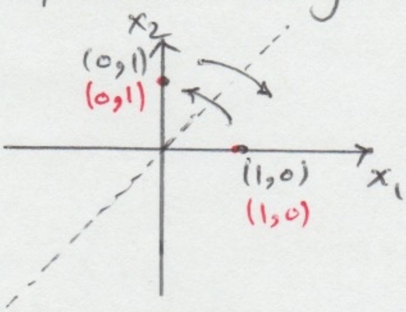
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2 - Reflection through x_2-axis



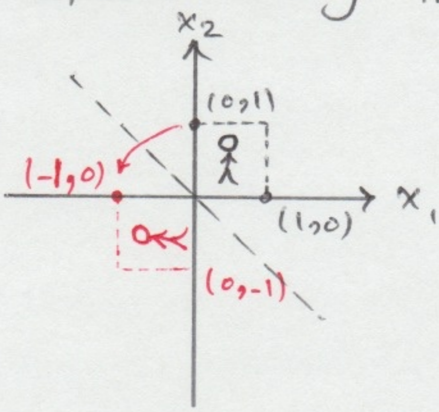
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 - Reflection through the line $x_1 = x_2$



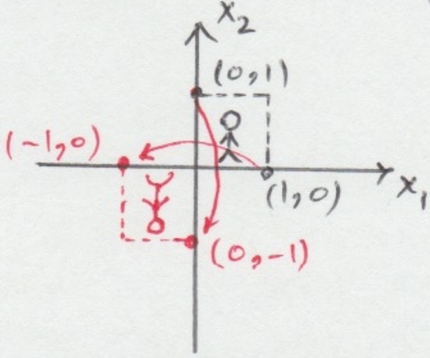
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

4 - Reflection through the line $x_1 = -x_2$



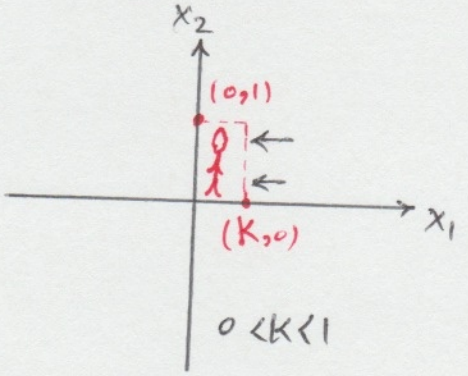
$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

5 - Reflection through the origin:

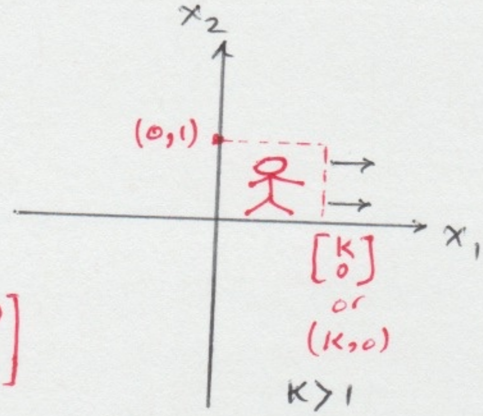


$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

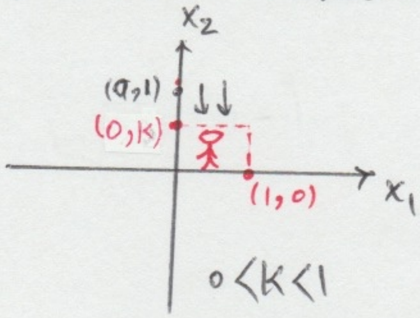
6- Horizonal contraction & expansion:



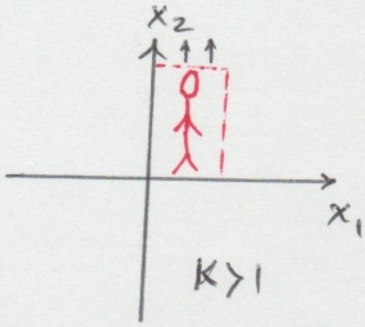
$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$



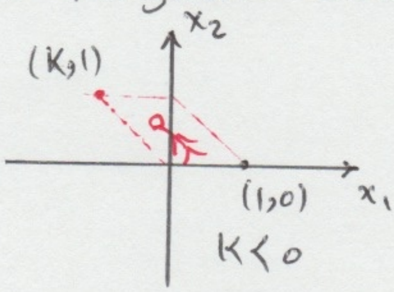
7- Vertical contraction and expansion:

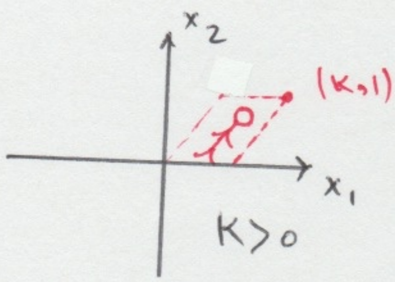


$$A = \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}$$



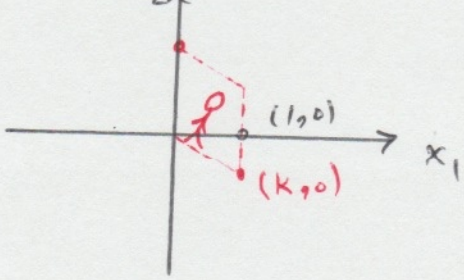
8- Horizontal shear:





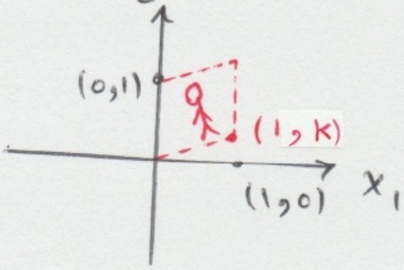
$$A = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}$$

9- Vertical shear

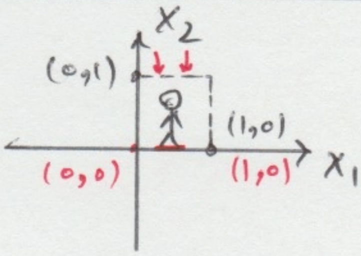


K < 0

$$A = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}$$



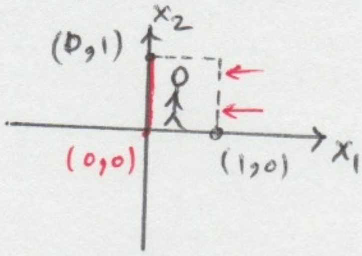
K > 0



10 - Projection onto the x_1-axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

11 - Projection onto the x_2-axis



$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition: A mapping $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be $\boxed{\text{onto}}$ $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.

* equivalently, we can say T is onto $\mathbb{R}^m$ if the range of T is all of the codomain $\mathbb{R}^m$

* We can also say T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if for each $\vec{b}$ in the codomain $\mathbb{R}^m$, there exist a solution for $T(\vec{x}) = \vec{b}$.

* This is really an existence question: Is there at least one solution such that $T(\vec{x}) = \vec{b}$ or $A\vec{x} = \vec{b}$?

* This means the mapping T is not onto when there is some $\vec{b}$ in the $\mathbb{R}^m$ for which the eq. $T(\vec{x}) = \vec{b}$ has no solution. (when $\vec{b}$ is not a linear combination of columns of A)

* Another way that we can look at this, is explained in the following Theorem:

Theorem: let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if & only if the columns of A span $\mathbb{R}^m$.

Definition: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$.

* Equivalently, T is one-to-one if, for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has either a unique solution or non at all. (This is really a uniqueness question)

* we can also say the mapping T is not one-to-one when some vector $\vec{b}$ in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$.

Ex. let T be a linear transformation whose standard matrix is $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? is T a one-to-one mapping?

solution: *Matrix A is already in echelon form. you can see that every row has a pivot. In section 1.4 we discussed a theorem that if there are pivots in each row of Matrix A, the system $A\vec{x} = \vec{b}$ is consistent (There is at least one solution). In other words, The linear transformation T is onto $\mathbb{R}^3$.

* the next question is if the transformation one-to-one? we see that matrix A has three pivot columns. In other words The matrix A has three basic variables and one free variable (x_3). Therefore, there exists more than one vector (infinitely many) that under transformation T, give rise to $\vec{b}$ in $\mathbb{R}^m$. In other words, each $\vec{b}$ is the image of more than one $\vec{x}$. So, T is NOT one-to-one.

The Matrix of a Linear Transformation

Theorem: let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then, T is one-to-one if & only if the eq. $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Note: I have included the proof of this theorem here just because it is beautiful and you get some good practice on how to prove a Theorem! However, I do not expect you to be able to do this proof in the test!

let's learn something first:

when we say P if & only if Q $(P \Leftrightarrow Q)$ or P iff Q, it means:

$$\begin{cases} \text{if } P \text{ is true, then } Q \text{ is true} & P \Rightarrow Q \\ \text{if } Q \text{ is true, then } P \text{ is true} & Q \Rightarrow P \end{cases}$$

This is equivalent to the following:

$$\begin{cases} \text{if } P \text{ is true, then } Q \text{ is true} & P \Rightarrow Q \\ \text{if } P \text{ is false, then } Q \text{ is false} & \sim P \Rightarrow \sim Q \end{cases}$$

In math 13 you will learn about different methods of proof. This method is called contrapositive reasoning.

For the proof of this Theorem we are going to use the second approach and prove:

1) if T is one-to-one, then $T(\vec{x}) = \vec{0}$ has a trivial solution

2) if T is not one-to-one, then $T(\vec{x}) = 0$ has more than one solution

Proof of Part 1) we know $T(\vec{0}) = \vec{0}$ because $A\vec{0} = \vec{0}$. If T is one-to-one, then $T(\vec{x}) = \vec{0}$ has at the most one solution and because $T(\vec{0}) = \vec{0}$, then that solution would be the trivial solution $\vec{x} = \vec{0}$

Proof of Part 2) if $T(\vec{x}) = \vec{0}$ is not one-to-one, Then there is a vector $\vec{b}$ That is the image of at least two different vectors in $\mathbb{R}^n$. let's call them $\vec{u}$ and $\vec{v}$.

$T(\vec{u}) = \vec{b}$ & $T(\vec{v}) = \vec{b}$. Therefore,

$$T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{b} - \vec{b} = \vec{0}$$

since $\vec{u}$ & $\vec{v}$ are two different vectors, $\vec{u}-\vec{v} \neq \vec{0}$ so, $T(\vec{x}) = \vec{0}$ has more Than one solution.

Theorem: let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be The standard matrix for T. Then, T is one-to-one if & only if The columns of A are linearly independent.

Proof: we just proved (in the previous Theorem) that T is one-to-one if & only if The eq. $T(\vec{x}) = \vec{0}$ has only trivial solution. This means $A\vec{x} = \vec{0}$ must have only trivial solution. We had a Theorem in 1.7 saying if the columns of matrix A are linearly independent, then The homogeneous system $A\vec{x} = \vec{0}$ has only a trivial solution!

(so we proved The Theorem)

Ex. let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show That T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

* $T(\vec{x}) = T(x_1, x_2) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Matrix A is 3×2 and is The standard matrix for the transformation. Columns of A are linearly independent because They are not multiples of each other. Since The Columns of matrix A are linearly independent, Then The transformation is one-to-one.

* The next question is if T is onto $\mathbb{R}^3$. Matrix A is 3x2 and has two columns. The columns of A span $\mathbb{R}^3$ if & only if A has three pivot positions (we learned This in section 1.4) since A has only two columns, Then columns of A do not span $\mathbb{R}^3$ so T is not onto $\mathbb{R}^3$.