In This section we define some important definitions and operations for matrices.

1) if we have an $m \times n$ matrix, like A, it means it has m rows & n columns. Elements (entries) of matrix A are shown by a_{ij}, which means the entry in i^{th} row and j^{th} column.

e.g. a_{23} is called $(2,3)$ entry and is The element (entry) in The second row & Third Column.

often, columns of a Matrix are shown by $\vec{a}_1, \vec{a}_2, \cdots$, and $\vec{a}_n$

for $\underbrace{m}_{\text{rows}} \times \underbrace{n}_{\text{Columns}} \longrightarrow A = [\vec{a}_1 \ \ \vec{a}_2 \ \cdots \ \vec{a}_n]$

$$A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & -2 \\ 5 & 0 & 6 & 1 \end{bmatrix} \Rightarrow A = [\vec{a}_1 \ \ \vec{a}_2 \ \ \vec{a}_3 \ \ \vec{a}_4]$$

$$a_1 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad a_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \quad a_4 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Diagonal entries of an $m \times n$ matrix:

$a_{11}, a_{22}, a_{33}, \ldots$ entries of a matrix are called diagonal entries. They form The main diagonal of a matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

Diagonal Matrix:

Is a <u>square</u> $n \times n$ matrix whose non-diagonal elements are zero.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Matrix Operation

Equal matrices: Two matrices are equal if they have the same size and the corresponding elements are equal.

Sum of Matrices:

If A & B are matrices with same size (m×n), Then The sum A+B is also an m×n matrix whose entries are the sum of corresponding entries of A & B.

$$A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

$$A+B=\begin{bmatrix} 4+1 & 0+1 & 5+1 \\ -1+3 & 3+5 & 2+7 \end{bmatrix}=\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

if A & B do not have the same size, then A+B is not defined.

scalar multiplication:

if r is a scalar & A is a matrix, then The scalar multiple rA is The Matrix whose entries are r times the Corresponding entries in A.

Ex. If $A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, and $B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, find $A-2B$.

$$A-2B=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}-2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}=\begin{bmatrix} 4-2 & 0-2 & 5-2 \\ -1-6 & 3-10 & 2-14 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Rules of sums & scalar multiplication of matrices:

let A, B, and C be matrices of the same size and let r and s be scalars:

$A+B=B+A$

$(A+B)+C=A+(B+C)$

$A+0=A$

$r(A+B)=rA+rB$

$(r+s)A=rA+sA$

$r(sA)=(rs)A$

Matrix Multiplication:

If A is an $m\times n$ matrix, and if B is an $n\times p$ matrix with columns $\vec{b}_1, \dots, \vec{b}_p$, then the product $A\times B$ is the $m\times p$ matrix whose columns are $A\vec{b}_1, \dots, A\vec{b}_p$. That is

$$AB = A\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p\end{bmatrix} = \begin{bmatrix}Ab_1 & Ab_2 & Ab_3 & \cdots & Ab_p\end{bmatrix}$$

This way of thinking about matrix multiplication is useful for Theoretical & practical applications.

* Very important: Multiplication of matrices corresponds to composition of linear transformations.

Ex. Compute AB where $A=\begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}$ and $B=\begin{bmatrix}4 & 3 & 6\\ 1 & -2 & 3\end{bmatrix}$

solution: we may write $B=\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix}$ and compute:

$$A\vec{b}_1 = \begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix} = \begin{bmatrix}11\\ -1\end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}3\\ -2\end{bmatrix} = \begin{bmatrix}0\\ 13\end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix}2 & 3\\ 1 & -5\end{bmatrix}\begin{bmatrix}6\\ 3\end{bmatrix} = \begin{bmatrix}21\\ -9\end{bmatrix}$$

$$\Longrightarrow AB = \begin{bmatrix}-11 & 0 & 21\\ -1 & 13 & -9\end{bmatrix}$$

(we calculate the linear combination of each row of A by each column of B)

* For the matrix multiplication to work, the number of columns of A should be the same as the number of rows of B.

E.x. A is an $m\times n$ matrix

B is an $n\times p$ matrix

AB is defined and is an $m\times p$ matrix

$$A_{m\times n}\ B_{n\times p} \rightarrow (AB)_{m\times p}$$

AB has the same number of rows as A and the same number of column as B.

Ex. if A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB & BA if they are defined?

A is 3×5

B is 5×2 $\longrightarrow$ AB is defined and is 3×2

BA is NOT defined.

* The following rule provides a very useful method for multiplication of AB.

Row - Column Rule for computing AB:

If the product AB is defined, then the entry in row i and column j of AB is the sum of products of corresponding entries from row $\underline{i}$ of A and column $\underline{j}$ of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m\times n$ matrix, then:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Ex. if $A=\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, find AB.

$$AB=\begin{bmatrix} 2\times4+3\times1 & 2\times3+3(-2) & 2\times6+3\times3 \\ 1\times4+(-5)\times1 & 1\times3+(-2)(-5) & 1\times6+(-5)\times3 \end{bmatrix}=\begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Ex. Find the entries of the second row of AB.

$$A=\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \quad B=\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} \rightsquigarrow \underset{1\times3}{\begin{bmatrix} -1 & 3 & -4 \end{bmatrix}}\underset{3\times2}{\begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}}=\underset{1\times2}{\begin{bmatrix} 5 & 1 \end{bmatrix}}$$

Properties of Matrix Multiplication:

let A be an $m\times n$ matrix, and let B and C have sizes for which the indicated sums & products are defined.

1) $A(BC)=(AB)C$ ⟶ Associative law of multiplication
2) $A(B+C)=AB+AC$ ⟶ left distributive law
3) $(B+C)A=BA+CA$ ⟶ right distributive law
4) $r(AB)=(rA)B=A(rB)$ (for any scalar r)
5) $I_m A = A = A I_n$ ⟶ Identity for matrix multiplication

Ex. let $A=\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$, show that these matrices do not commute. That is, verify that $AB \neq BA$.

$$AB=\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}=\begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

As it can be seen, $AB \neq BA$

$$BA=\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}=\begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

Example: let $A=\begin{bmatrix}2 & -3\\ -4 & 6\end{bmatrix}$, $B=\begin{bmatrix}8 & 4\\ 5 & 5\end{bmatrix}$, and $C=\begin{bmatrix}5 & -2\\ 3 & 1\end{bmatrix}$, verify $AB=AC$ and yet $C\neq B$.

$$AB=\begin{bmatrix}2 & -3\\ -4 & 6\end{bmatrix}\begin{bmatrix}8 & 4\\ 5 & 5\end{bmatrix}=\begin{bmatrix}1 & -7\\ -2 & 14\end{bmatrix}$$

$AB=AC$ But $B\neq C$

$$AC=\begin{bmatrix}2 & -3\\ -4 & 6\end{bmatrix}\begin{bmatrix}5 & -2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}1 & -7\\ -2 & 14\end{bmatrix}$$

Be Careful (warnings):

1/ In general $AB\neq BA$

2/ The cancellation laws do not hold for matrix multiplication. that is, if $AB=AC$, Then it is not true in general that $B=C$.

3/ If a product AB is the zero matrix, you can not conclude in general that either $A=0$ or $B=0$.

The Transpose of a Matrix:

Given an $m\times n$ matrix A, the transpose of A is the $n\times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

e.g. $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \overset{iff}{\Longleftrightarrow} A^T=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$, $B=\begin{bmatrix}-5 & 2\\ 1 & -3\\ 0 & 4\end{bmatrix} \longrightarrow B^T=\begin{bmatrix}-5 & 1 & 0\\ 2 & -3 & 4\end{bmatrix}$

$C=\begin{bmatrix}1 & 1 & 1 & 1\\ -3 & 5 & -2 & 7\end{bmatrix} \rightsquigarrow C^T=\begin{bmatrix}1 & -3\\ 1 & 5\\ 1 & -2\\ 1 & 7\end{bmatrix}$

Properties of Transpose of a Matrix:

let A & B denote matrices whose size are appropriate for the following sums & products.

1) $(A^T)^T=A$

2) $(A+B)^T=A^T+B^T$

3) for any scalar r, $(rA)^T=rA^T$

4) $(AB)^T=B^TA^T$

↳ The Transpose of a product of matrices equals the product of their transpose in reverse order