The inverse of A Matrix

In This section, we investigate the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

recall $5^{-1} \cdot 5 = 1$ & $5 \cdot 5^{-1} = 1$.

In matrices, an $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such That $CA = I_{n\times n}$ and $AC = I_{n\times n}$.

matrix C in This case is the inverse of matrix A. C is uniquely determined by A. This unique inverse is denoted by A^{-1}. so:

$$AA^{-1} = A^{-1}A = I$$

Def.: A matrix that is not invertible is sometimes called a singular matrix and invertible matrix is called a nonsingular matrix.

Ex. If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, show that $C = A^{-1}$.

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow A^{-1} = C$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -14+15 & -35+35 \\ 6-6 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, Then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, Then A is not invertible.

* $ad - bc$ is called The determinant of A.

$$\det A = ad - bc$$

Ex. Find inverse of $A=\begin{bmatrix}3&4\\5&6\end{bmatrix}$.

$\det A = 18 - 20 = -2 \longrightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix}6&-4\\-5&3\end{bmatrix}$

Theorem:

If A is an invertible $n\times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the eq. $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$.

(This formula in reality is never used to solve systems of eqs. as row reduction of the augmented matrix $[A\ b]$ is almost always faster).

Ex. Use the inverse of the matrix A in the previous example to solve the system:

$$\begin{cases}3x_1+4x_2=3\\5x_1+6x_2=7\end{cases} \longrightarrow A\vec{x}=\vec{b} \Rightarrow \begin{bmatrix}3&4\\5&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}$$

$$\Rightarrow \vec{X} = -\tfrac{1}{2}\begin{bmatrix}6&-4\\-5&3\end{bmatrix}\begin{bmatrix}3\\7\end{bmatrix} \Rightarrow \vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix} = -\tfrac{1}{2}\begin{bmatrix}18-28\\-15+21\end{bmatrix}$$

$$= -\tfrac{1}{2}\begin{bmatrix}-10\\+6\end{bmatrix} = \begin{bmatrix}5\\-3\end{bmatrix}$$

Theorem: * If A is an invertible matrix, then A^{-1} is invertible &

$(A^{-1})^{-1} = A$

* If A & B are $n\times n$ invertible matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

* If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is:

$(A^T)^{-1} = (A^{-1})^T$

Elementary Matrices:

An elementary matrix is a matrix that is obtained by performing a single elementary row operation on an identity matrix.

* The rows of an identify matrix can be multiplied by a number.
* Multiple of a row may be added to another row
* a row may be replaced by another row.

Ex. let $E_1=\begin{bmatrix}1&0&0\\0&1&0\\-4&0&1\end{bmatrix}$, $E_2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}$, and $A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$.

Compute E_1A, E_2A, & E_3A.

$$E_1A=\begin{bmatrix}a&b&c\\d&e&f\\g-4a&h-4b&i-4c\end{bmatrix}\quad E_2A=\begin{bmatrix}d&e&f\\a&b&c\\g&h&i\end{bmatrix}\quad E_3A=\begin{bmatrix}a&b&c\\d&e&f\\5g&5h&5i\end{bmatrix}$$

The take home message from the previous example is:

* If an elementary operation is performed on an $m\times n$ matrix A, The resulting matrix can be written as EA, where the $m\times m$ matrix is created by performing the same row operation on I_m.

* Important * Each elementry matrix is invertible. the inverse of E is the elementary matrix of the same type That transforms E back to I.

Ex. Find The inverse of $E_1=\begin{bmatrix}1&0&0\\0&1&0\\-4&0&1\end{bmatrix}$

To transform E_1 to I, we have added (4 times row 1) to row 3. The elementary matrix that does this is $\begin{bmatrix}1&0&0\\0&1&0\\+4&0&1\end{bmatrix}$

Notice the difference in sign.

Very important Theorem:

An $n \times n$ matrix A is invertible iff (if and only if) A is row equivalent to I_n. It means, any sequence of elementary row operations that reduces A to I_n, transforms I_n into A^{-1}.

An Algorithm for finding A^{-1}:

The previous theorem provides a very useful technique to find A^{-1} for a matrix. If we form the augmented matrix $[A\ I]$ and by row operations we convert A to I, the same row operations will convert I to A^{-1} if the matrix A is invertible.

Ex. Find the inverse of matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

You can double check your work by calculating AA^{-1} to see if you obtain I