Characterization of Invertible Matrices

This section provides a review of most of the concepts introduced in chapter 1, in relation to a system of n linear equations in n unknown and to square matrices.

* The Invertible Matrix Theorem: (only for square matrices)

let A be a square $n \times n$ matrix, then the following statements are equivalent. That is, for a given A the following statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to $n \times n$ identity matrix.

c. A has n pivot positions

d. The eq. $A\vec{x} = \vec{0}$ has only the trivial solution.

e. The columns of A for a linearly independent set.

f. The linear transformation $x \longrightarrow A\vec{x}$ is one-to-one

g. The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j. There is an $n \times n$ matrix C such that $CA = I$

k. There is an $n \times n$ matrix D such that $AD = I$

l. A^T is an invertible matrix.

Ex. Use the Invertible Matrix Theorem to decide if A is invertible.

$$A=\begin{bmatrix}1 & 0 & -2\\ 3 & 1 & -2\\ -5 & -1 & 9\end{bmatrix} \xrightarrow[\text{Reduced}]{\text{Row}} \begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 4\\ 0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 4\\ 0 & 0 & 3\end{bmatrix}$$

so A has (3) pivot positions & hence is invertible.

* The Invertible Matrix Theorem divides the set of all $n\times n$ matrices into two disjoint classes: invertible (nonsingular) matrices & noninvertible (singular) matrices.

* The negation of a statement in the Theorem, describes a property of every singular matrix.

  for example, an $n\times n$ singular matrix is not row equivalent to I_n, does not have n pivot positions, and has linearly dependent columns.

* It should be emphasized that the Invertible Matrix theorem applies only to square matrices.

Invertible Linear Transformations:

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there exists a function (transformation) $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

$T(S(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

(we call S inverse of T and we show it with T^{-1})

in other words

$$T^{-1}(T(\vec{x})) = \vec{x}$$

$$T(T^{-1}(\vec{x})) = \vec{x}$$

Theorem: let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T.

Then T is invertible if & only if A is an invertible matrix. In that case the linear transformation S or T^{-1} is given by $S(\vec{x}) = T^{-1}(x) = A^{-1}\vec{x}$ is the unique function (transformation) satisfying

$$T^{-1}(T(\vec{x})) = \vec{x} \quad \& \quad T(T^{-1}(\vec{x})) = \vec{x}$$

* This means if T is a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, the columns of Matrix A of T are linearly independent. This means A is invertible, by the Invertible Matrix Theorem and T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. It also means T is invertible.