This section focuses on important sets of vectors in $\mathbb{R}^n$ called subspaces. Subspaces provide useful information about the equation $A\vec{x} = \vec{b}$.

Def. of a Subspace:

A subspace of $\mathbb{R}^n$ is any set like H in $\mathbb{R}^n$ that has three properties:

1- The zero vector $\vec{0}$ is in H

2- For each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u}+\vec{v}$ is in H.

3- For each $\vec{u}$ in H and each scalar c, the vector $c\vec{u}$ is in $\vec{H}$.

in other words, "A subspace is closed under addition & scalar multiplication"

Ex. If $\vec{v}_1$ and $\vec{v}_2$ are in $\mathbb{R}^n$ and $H = \text{Span}\{v_1, v_2\}$, then H is a subspace of $\mathbb{R}^n$. (prove)

We remember that span$\{v_1, v_2\}$ means the set of all the vectors that can be produced by the linear combination of vectors $\vec{v}_1, \vec{v}_2$.

e.g. $\vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2$

$\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2$ $\quad$ (s_1, s_2, t_1, t_2 being scalars)

1/ it is obvious that $\vec{0}$ (zero vector) is in H.

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 \quad \text{(a linear combination of } \vec{v}_1 \text{ \& } \vec{v}_2\text{)}$$

2/ Assuming the two arbitrary vectors $\vec{u}$ and $\vec{v}$ are in H, is $\vec{u}+\vec{v}$ also in H?

$u+v = (s_1+t_1)\vec{v}_1 + (s_2+t_2)\vec{v}_2$. This is another linear combination of $\vec{v}_1$ and $\vec{v}_2$ so it must be in H.

3/ Assuming the arbitrary vector $\vec{u}$ is in H, is $c\vec{u}$ also in H?

$$c\vec{u} = c(s_1\vec{v}_1 + s_2\vec{v}_2) = (cs_1)\vec{v}_1 + (cs_2)\vec{v}_2$$

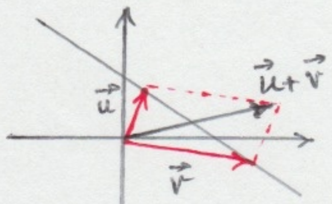
This is another linear combination of $\vec{v}_1$ & $\vec{v}_2$, so it must be in H.

Therefore, H is a subspace.

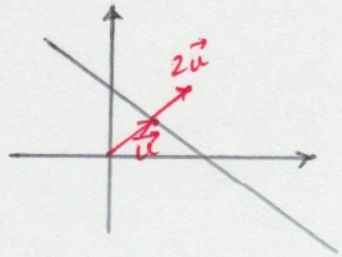
Ex. Show that a line ℓ, that does not pass through the origin is not a subspace.

1/ The line does not pass through the origin, so it does not include the origin (the zero vector)

2/ for each $\vec{u}$ & $\vec{v}$ on the line, $\vec{u}+\vec{v}$ is not on L.



3/ for each $\vec{u}$ on the line, $c\vec{u}$ is not on the line.



* For $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p$ or the span$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a subspace. We can refer to span$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ as the subspace spanned or generated by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Definition of a column space:

The Column space of a Matrix A is the set Col A of all linear combinations of the Columns of A.

* If $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$, with the Columns in $\mathbb{R}^m$ (each vector has m members). The Col A is a subspace of $\mathbb{R}^m$.

Ex. let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ be vector $\vec{b}$. Determine whether $\vec{b}$ is in the Column space of A.

This is possible only if vector $\vec{b}$ can be written down as the linear combination of column A. It means $\vec{b}$ is in column space of A if and only if $A\vec{x} = \vec{b}$ has a solution.

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 2 & 6 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This eq. is consistent and $\vec{b}$ is in Col A.

Definition of Null space:

The null space of a Matrix A is the set Nul A of all Solutions of the homogeneous equation $A\vec{x} = \vec{0}$

Basis for a Subspace:

Typically, a subspace contains an infinite number of vectors. It is better to have a small finite set that span the whole subspace. The smaller the set, the better.

Definition: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

* The columns of an invertible $n\times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

One such matrix is the $n\times n$ identity matrix. Its columns are denoted by $e_1, \dots, e_n$

$$e_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\quad e_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\quad e_3=\begin{bmatrix}0\\0\\1\\0\\\vdots\\0\end{bmatrix},\quad \text{and} \dots \quad e_n=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}.$$ The set

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is called The Standard basis for $\mathbb{R}^n$.

*** The following example shows the standard procedure for writing the solution set of $A\vec{x}=\vec{0}$ in parametric vector form actually identifies a basis for Nul A.

Ex. Find a basis for the null space of the following matrix.

$$A=\begin{bmatrix}-3 & 6 & -1 & 1 & -7\\ 1 & -2 & 2 & 3 & -1\\ 2 & -4 & 5 & 8 & -4\end{bmatrix}$$

Solution: First we write the augmented matrix for system $A\vec{x}=\vec{0}$ and we row reduce the matrix.

$$[A\ \ 0] \sim \begin{bmatrix}1 & -2 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 2 & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0\\ x_3 + 2x_4 - 2x_5 &= 0\\ 0 &= 0\end{aligned}$$

x_2, x_4, and x_5 are free variables:

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2x_2+x_4-3x_5\\x_2\\-2x_4+2x_5\\x_4\\x_5\end{bmatrix}=x_2\underbrace{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}}_{\vec{u}}+x_4\underbrace{\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}}_{\vec{v}}+x_5\underbrace{\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}}_{\vec{w}}=x_2\vec{u}+x_4\vec{v}+x_5\vec{w}$$

as we saw in the solution, Nul A coincides with the set of all linear combinations of $\vec{u}$, $\vec{v}$, and $\vec{w}$, that is $\{\vec{u}, \vec{v}, \vec{w}\}$ generate Nul A. In fact, this construction of $\vec{u}$, $\vec{v}$, and $\vec{w}$ makes them linearly independent, because $x_2\vec{u} + x_4\vec{v} + x_5\vec{w} = \vec{0}$ only if weights x_2, x_4 and x_5 are all zero (think about the entries 2, 4, and 5 in the vector $x_2\vec{u} + x_4\vec{v} + x_5\vec{w}$). This means $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for Nul A.

Ex. Find a basis for the column space of the matrix $\begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
Pay attention that in this example, we are trying to find the basis for column space (unlike the previous problem that we found basis for Null space).

Do NOT confuse these two!

let's call this matrix B and its columns $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_5$.

we can see that $\vec{b}_3 = -3\vec{b}_1 + 2\vec{b}_2$ & $\vec{b}_4 = 5\vec{b}_1 - \vec{b}_2$

now if $\vec{v}$ is any vector in Col B (linear combination of columns)

Then $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5$

↳ by replacing $\vec{b}_3$ and $\vec{b}_4$ by $-3\vec{b}_1 + 2\vec{b}_2$ and $5\vec{b}_1 - \vec{b}_2$ respectively, we see that $\vec{v}$ is a linear combination of $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_5$ → This means $\{\vec{b}_1, \vec{b}_2, \vec{b}_5\}$ spans Col B.

* We also know that $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_5$ are linearly independent because they are columns from an identity Matrix.

* This means, the pivot columns of B form a basis for B

* **Theorem:** pivot columns of matrix A form a basis for the column space of A.

Warning: Be careful to use pivot columns of A itself to the basis of Col A. The columns of an echelon form of matrix A are often not in the column space of A.

* in other words, you can row reduce the matrix A to find the pivot columns. Those pivot columns for reduced echelon form of A are <u>NOT</u> the basis for Col A. However, since a matrix A is equivalent to its reduced form, the pivot columns of reduced form of A are also the pivot columns of A. Those columns in A will be the basis for Col A.

$$\text{e.g. } A=\begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix} \xrightarrow[\text{echelon form}]{\text{reduced}} \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 2, and 5 are pivot columns, therefore, $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ are the basis for reduced echelon matrix column space

This means that columns 1, 2, and 5 are also pivot columns for matrix A (non-reduced)

This means the vectors $\begin{bmatrix}1\\-2\\2\\3\end{bmatrix}, \begin{bmatrix}3\\-2\\3\\4\end{bmatrix}$, and $\begin{bmatrix}-9\\2\\1\\8\end{bmatrix}$ are the basis for the column space of matrix A or Col A.