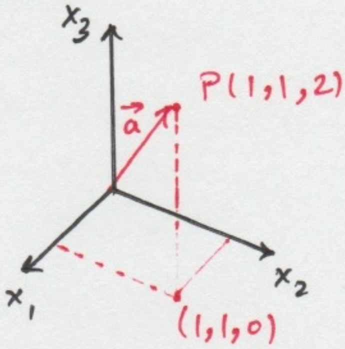
suppose that we have coordinates of a point in 3D coordinate system. For instance, $P(1,1,2)$. we know that this point is associated with a position vector, say $\vec{a}$. The components of vector $\vec{a}$ are the coordinates of point P.

$\vec{a} = \langle 1, 1, 2 \rangle$, or in linear algebra we show this as $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.



it can be seen that the coordinates of point p and hence the components of vector $\vec{a}$ may be written as the linear combination of the standard basis vectors $\vec{i}, \vec{j}, \vec{k}$ such that

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{a} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The weights ($c_1 = 1$, $c_2 = 1$, and $c_3 = 2$) are called the coordinates of point p (or vector $\vec{a}$). Remember that $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Definition: Suppose the set $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p\}$ is a basis for subspace H. for each $\vec{x}$ in H, the coordinates of $\vec{x}$ relative to the basis B are the weights $c_1, \dots, c_p$ such that $\vec{x} = c_1\vec{b} + c_2\vec{b} + \dots + c_p\vec{b}$.

The vector $[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$ is called the coordinate vector of $\vec{x}$ (relative to B) or the coordinate vector of $\vec{x}$.

Example: let $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $B = \{v_1, v_2\}$. Determine if $\vec{x}$ is in H, and if it is, find the coordinate vector of $\vec{x}$ relative to B.

B is a basis for $H = \text{Span}\{v_1, v_2\}$ because v_1 & v_2 are linearly independent. if $\vec{x}$ is in H, then the following system is consistent. (The augmented matrix)

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 3 \\ 0 & 2 & 6 \\ 0 & 5/3 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 1/3 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $c_1 = 2$, $c_2 = 3$, $[X]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. So the basis B determines a coordinate system on H.

The dimension of a subspace:

Def.: The dimension of a nonzero subspace, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of a zero subspace $\{\vec{0}\}$ is defined to be zero.

* Based on this definition, to find the dimension of $\text{Nul } A$, we just identify and count the free variables of $A\vec{x} = \vec{0}$.

Def.: The rank of a Matrix A, denoted by $\text{rank } A$, is the dimension of the column space of A.

* Based on this definition, the pivot columns of A form a basis for $\text{Col } A$, then the rank of A is just the number of pivot columns in A.

Ex. Determine the rank of the following matrix.

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & -10 & 15 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad\quad \uparrow$

pivot Columns

Therefore, the rank of matrix A or rank A is 3.

* We know that nonpivot columns correspond to the free variables in $A\vec{x} = \vec{0}$. We also know that the number of pivot columns is associated with the rank of Matrix A. Then, the number of pivot columns (rank of A) plus the number of nonpivot columns (the dimension of nul A) is exactly the number of columns of A. We can find the following useful connection between the dimension of Col A and the dimension of Nul A.

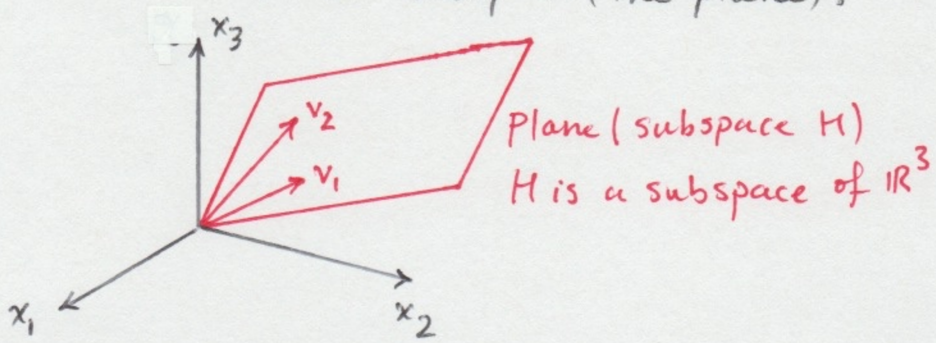
→ it is called the rank Theorem

The rank Theorem:

if a matrix A has n columns, then $\text{rank } A + \dim \text{Nul } A = n$

The basis Theorem (Example)

Think about a two dimensional plane in $\mathbb{R}^3$ that passes through the origin. If we pick two linearly independent vectors on that plane, these two vectors are automatically a basis for the plane. Also, any set of two vectors in that plane that span the plane is automatically a basis of the subspace (the plane).



v_1 and v_2 are the basis for H (the plane) and the two vectors that we pick in the plane, if they are linearly independent, they span H.

The basis Theorem (statement)

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

The Invertible Matrix Theorem (continued)

Now that we have learned about the Col A, Nul A, rank A, dim Nul A, and ..., we can add the following extension to the Invertible Matrix Theorem:

Theorem: let A be an $n \times n$ matrix, then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$

n. Col $A = \mathbb{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\vec{0}\}$

r. dim Nul $A = 0$