Introduction to determinants

Previously, we discussed the determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc$$

e.g. if $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \rightarrow \det A = 4 - 3 = 1 \rightarrow \det A = 1$

A lot of times, we need to find the determinant of a larger matrix like a 3x3 or 4x4 (or in general an nxn matrix)

The process to find the determinant of larger nxn matrices is as followed:

1- We pick the first element in the first row of the matrix. (later, we will expand this technique to any row or column).

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

2- Eliminate all the elements in the same row & column associated with the element that we have picked. This gives rise to another matrix that is $(n-1)(n-1)$ instead of $n \times n$.

$$\begin{bmatrix} 1 & \cancel{5} & \cancel{0} \\ \cancel{2} & 4 & -1 \\ \cancel{0} & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}$$

3- Multiply the element a_{11} by the determinant of the matrix that is left from the process in step 2.

$$1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} = 1(-2) = -2$$

These numbers are called Cofactors, and denoted in general by C_{ij}.
e.g. $C_{11} = -2$

* Note: The determinant of the matrix and its sign (the sign alternates) is considered the Cofactor of the (i, j) element.

4- Pick the second element in the first row of the original matrix (a_{12}) and eliminate the elements in the first row and the second column of the matrix to create another $(n-1)(n-1)$ matrix.

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

5- Multiply a_{12} with a negative sign by the determinant of the new matrix.

$$-5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} = 2(0) = 0$$

$\hookrightarrow C_{12} = 0$

6- Continue the process by picking the remaining elements in the first row following steps 2 to 5.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = 0(-4) = 0$$

$C_{12} = -4$

7- Add all the final results for steps 3, 5, and ... to find det A.

det $A = -2 - 0 + 0 = -2$, Therefore, the determinant of the 3×3 matrix A is equal to -2.

Important Note: It is very important to change the sign every other element to calculate det A. In other words, the sign of a_{11} is positive, a_{12} is negative, a_{13} is positive and in general the sign of a_{ij} is $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

Important:

This technique is called cofactor expansion. The cofactor expansion does not necessarily need to be done across the first row. This process may be done across any row or any column.

* The (i,j)-cofactor of A is the number C_{ij} such that $C_{ij} = (-1)^{i+j} \det A_{ij}$. These are the numbers that we got from steps 2, 5, ...

Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The expansion across the i^{th} row using cofactors lead to:

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + a_{i3} C_{i3} + \cdots + a_{in} C_{in}$$

The cofactor expansion down the j^{th} column is:

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + a_{3j} C_{3j} + \cdots + a_{nj} C_{nj}$$

Ex. Use expansion across the third row to compute $\det A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + (0) \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 0(-5) + 2(-1-0) + 0(4-10)$$

$$= -2$$

* This theorem is very helfpul finding determinant of matrices That contain many zeros. We can reduce our calculation load by choosing the rows/columns that have The max. number of zeros.

Ex. Compute det A where $A=\begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

$$\det A = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3\,(2)\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 6\left(1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} \right) = 6\left(1\times(-2) - 5(0) + 0(-4)\right)$$

$$= -12$$

* The matrix in this example was nearly triangular (it was <u>NOT</u> exactly triangular). The method used for finding the determinant may be adopted to prove the following theorem:

Theorem: If a is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Ex. Find the determinant of matrix $B=\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

The matrix is triangular, therefore,

det $A = 1\times1\times(-3)(-2) = 6$