In this section we want to investigate how determinant of a matrix is affected when row operations are applied to matrices.

Ex. In the following example, we apply different elementary row operations on matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to see how the determinant of the matrix changes.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \det A = ad - bc$

if we switch the rows $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \longrightarrow \det B = bc - ad \Rightarrow \det B = -\det A$

if we multiply row 1 with a constant & subtract it from another row.

$$B = \begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix} \Rightarrow \det B = a(d-kb) - b(c-ka)$$

$$= ad - \cancel{kab} - bc + \cancel{kab} = ad - bc$$

$$\Rightarrow \det B = \det A$$

if we multiply a row with a constant k:

$B = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix} \Rightarrow \det B = kad - kbc = k(ad - bc) \Rightarrow \det B = k \det A$

<u>Theorem</u>: Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce matrix B, then $\det B = \det A$

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$

c. If one row of A is multiplied by k to produce B, then $\det B = k \det A$

Ex. compute det A, where $A=\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -1\,(3)(-5) = 15$$

Ex. Compute det A, where $A=\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1) = -36$$

Note: Although the echelon form of a matrix A is not unique (reduced echelon form is unique but the echelon form is not unique), and also pivots are not unique, the product of the pivots is unique. (we still have to consider the sign change when we exchange the rows. we should also consider multiplication by a constant if we factor out a constant from a row).

* Note: we can add another statement to the Invertible Matrix Theorem.

Theorem: A Matrix $\underline{A}$ is invertible if & only if $\det A \neq 0$.

Note: A useful corollary of this theorem is that $\det A = 0$ when the columns of A are linearly dependent.

Note: We know that columns of A are rows of A^T. By the Invertible Matrix Theorem, if A^T is singular ($\det A^T = 0$), then A is singular. (Go back to the statement of Invertible Matrix Theorem in 2.3).

In other words, if the rows (This is true about the columns of A as well) of Matrix A are linearly dependent, then the matrix A is singular & $\det A = 0$.

Ex. Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

If we add two times row one to row 3 ($2R + R_3$), we will have

$$\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

because the second row & the third row of this matrix are equal, the rows are linearly dependent, therefore, $\det A = 0$.

In practice, it is not easy to see the linear dependence of the rows or columns before we start the row reduction process)

Ex. Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

we add the second row to the fourth row:

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2(-5) \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -2(5)(-3) = -30$$

Column operation: The next theorem shows that column operations have the same effect on determinants as row operations.

Theorem: If A is an $n\times n$ matrix, then $\det A^T = \det A$

Remark: In order to prove this Theorem, we use the principal of Mathematical Induction. It is a method that is used widely in mathematical proof.

* let $P(n)$ be a mathematical statement that is either true or false. By showing $P(1)$ is true (for the value of $n=1$, the statement is true) and then assuming $P(k)$ is true (for the value of $n=k$ the statement is true). If we can prove $P(k+1)$ is true, then $P(n)$ is true for any value of n.

Proof: This theorem is obvious for $n=1$ (for 1×1 matrix). Suppose the Theorem is true for $n=k$. Now, let's assume that A is a $(k+1)(k+1)$ matrix. The cofactor expansion of $\det A$ along the first row equals the cofactor expansion of $\det A^T$ down the first column. This means, showing the theorem is true for $n=1$ and assuming the Theorem is correct for $n=k$, we were able to prove the theorem is true for $n=k+1$.

$n=1$: $A=[a]$, $A^T=[a]$, $\det A = \det A^T$

$$n=k: \; A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}, \quad A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & & a_{k2} \\ \vdots & \vdots & & \vdots \\ a_{1k} & a_{2k} & & a_{kk} \end{bmatrix}$$

let's assume $\det A = \det A^T$

Now, let's see if we can prove the Theorem is correct for $n=K+1$.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ \vdots & & & \\ a_{K1} & a_{K2} & \cdots & a_{KK} \end{bmatrix}$$

$$\Rightarrow \det A = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2K} \\ a_{32} & a_{33} & \cdots & a_{3K} \\ \vdots & & & \\ a_{K2} & a_{K3} & \cdots & a_{KK} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2K} \\ a_{31} & a_{33} & \cdots & a_{3K} \\ \vdots & & & \\ a_{K1} & a_{K3} & \cdots & a_{KK} \end{vmatrix} + \cdots \quad (\mathrm{I})$$

$$A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{K1} \\ a_{12} & a_{22} & & a_{K2} \\ \vdots & \vdots & & \vdots \\ a_{1K} & a_{K2} & & a_{KK} \end{bmatrix}$$

$$\Rightarrow \det A^T = a_{11}\begin{vmatrix} a_{22} & a_{32} & \cdots & a_{K2} \\ a_{23} & a_{33} & \cdots & a_{K3} \\ \vdots & & & \\ a_{K3} & a_{K4} & \cdots & a_{KK} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{31} & \cdots & a_{K1} \\ a_{23} & a_{33} & \cdots & a_{K3} \\ \vdots & & & \\ a_{2K} & a_{3K} & \cdots & a_{KK} \end{vmatrix} + \cdots \quad (\mathrm{II})$$

The determinants of $K\times K$ matrices are the same (we assumed it for $K\times K$ matrices).

Therefore, the $\det A$ in (I) is the same as the $\det A^T$ in (II)

Theorem: If A and B are $n\times n$ matrices, then $\det(AB)=\det A \cdot \det B$

Ex. verify the Theorem for $A=\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$\det A = 12-3=9$

$\det B = 8-3=5$

$\Rightarrow \det A \cdot \det B = 45$

$$\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix} = 25\cdot 13 - 20\cdot 14 = 45$$