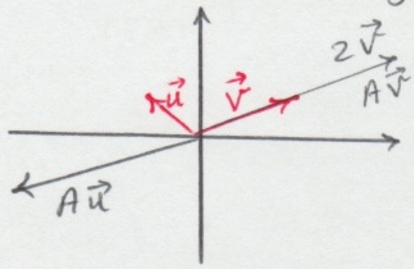
Let's think about transformation T, with its standard matrix A defined as $\vec{x} \longrightarrow A\vec{x}$.

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let's find the images of $\vec{u}$ and $\vec{v}$ and sketch them.



$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow$ in this case A only stretches or dialates vector $\vec{v}$.

This is a special vector called <u>eigenvector</u>

This is a special value called <u>eigenvalue</u>

Def.: An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called eigenvalue of A, if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$. Such $\vec{x}$ is called an eigenvector corresponding to λ.

Ex. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \vec{u}$, and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\vec{u}$ & $\vec{v}$ eigenvectors of A?

$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$

$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of A and corresponds to -4

Ex. Show that $\lambda = 7$ is an eigen value of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

If $\lambda = 7$ is an eigenvalue of A, Then

$$A\vec{x} = 7\vec{x} \quad \text{or} \quad A\vec{x} - 7\vec{x} = \vec{0}$$

$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$ The columns are linearly dependent. Therefore, we have nontrivial solution. Let's find the solutions of This system.

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

x_1 is a basic variable and x_2 is a free variable.

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ each vector of This form w/ $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$

warning: Although row reduction was used in the previous example to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A is usually does not display the eigenvalues of A.

Note: λ is an eigenvalue of $n \times n$ matrix if & only if The eq. $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution.

The set of all The solutions of This eq. is The null space of matrix $A - \lambda I$. This set is a subspace of $\mathbb{R}^n$ and is called The eigenspace of A corresponding to λ.

Ex. let $A=\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

$(A-\lambda I)\vec{x}=\vec{0}$ works for $\lambda=2$

$$A-\lambda I=\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{basic} & \text{Free} & \text{Free} \end{array} \end{array} \Rightarrow \begin{array}{l} 2x_1 - x_2 + 6x_3 = 0 \\ x_1 = \frac{1}{2}x_2 - 3x_3 \end{array}$$

$$\Rightarrow \vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix}+\begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigenspace is a two dimensional subspace of $\mathbb{R}^3$.

basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Theorem: The eigenvalues of a triangular matrix are the entries on its main diagonal.

for example, if $A=\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$

the eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 & 1.

Proof: for simplicity, let's consider a 3x3 matrix.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

λ is the eigenvalue of A if & only if $(A-\lambda I)\vec{x}=\vec{0}$ has a nontrivial solution, which means if there is a free variable and that only happens when one of the entries on the diagonal is zero (one of the columns become a non-pivot). This happens when $\lambda=a_{11}$ or $\lambda=a_{22}$ or $\lambda=a_{33}$.

Note: what is the meaning of $\underline{0}$ being an eigenvalue?

* It means $A\vec{x} = 0\vec{x} = \vec{0}$, Therefore, $A\vec{x} = \vec{0}$ and has nontrivial solution. This only happens when n is not invertible. In other words, if A was invertible, we could not have a nontrivial solution for $A\vec{x} = \vec{0}$ (According to The Invertible Matrix Theorem)

* Theorem: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of an $n \times n$ matrix A, Then The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ are linearly independent.