useful information about the eigenvalues of a square matrix is enclosed in a special scalar equation called characteristic equation of A. Let's start with a simple example.

Ex. Find the eigenvalues of $A=\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

We must find all the scalar λ such that $(A-\lambda I)\vec{x}=\vec{0}$ has nontrivial solution. By the Invertible Matrix Theorem, we know that this problem is equivalent to finding all λ values such that the matrix $A-\lambda I$ is not invertible, in other words, $\det(A-\lambda I)=0$

$$\Rightarrow \quad A-\lambda I=\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}, \quad \det(A-\lambda I)=(2-\lambda)(-6-\lambda)-9=0$$

$$\Rightarrow -12-2\lambda+6\lambda+\lambda^2-9=0$$

$$\Rightarrow \lambda^2+4\lambda-21=0$$

$$(\lambda-3)(\lambda+7)=0 \quad \begin{matrix} \nearrow \lambda=3 \\ \searrow \lambda=-7 \end{matrix}$$

The eigen values of this matrix are 3 & -7.

The equation $\lambda^2+4\lambda-21=0$ is called the characteristic Equation of Matrix A.

* Let's Revist The Invertible Matrix Theorem:

previously, we discussed that if A is an $n\times n$ matrix, then A is an invertible matrix iff the $\det A\neq 0$ (determinant of A is not zero). We also know that if matrix A is invertible and A is reduced to matrix U in echelon form without scaling (only by row interchange & row replacement), then:

$\det A=(-1)^r\cdot$(product of pivots in U) $\rightsquigarrow$ r is the # of row interchanges

The reason is that the determinant of any triangular matrix is the product of entries in main diagonal.

We also know for a triangular matrix the entries on the diagonal are the eigenvalues. As a result, we can add the following sections to the Invertible Matrix Theorem:

The Invertible Matrix Theorem:

let A be an $n\times n$ matrix. Then A is invertible iff:

s. The number 0 is not an eigenvalue of A.

t. The determinant of A is nonzero $\det A \neq 0$

Reminder (properties of determinants):

Let A & B be $n\times n$ matrices

a. A is invertible iff $\det A \neq 0$

b. $\det(AB) = (\det A)(\det B)$

c. $\det A^T = \det A$

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange, changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor

The characteristic Equation:

Def. A scalar λ is an eigenvalue of $n\times n$ matrix A if λ satisfies the characteristic eq. $\det(A-\lambda I)=0$

Ex. Find the characteristic equation of $A=\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$A-\lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$\det(A-\lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0$ → $\lambda = 5$, $\lambda = 5$, $\lambda = 1$, $\lambda = 3$

by expanding this product, we can also write $\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

This is the characteristic eq. of matrix A.

Note: It can be shown that if A is an $n\times n$ matrix, then $\det(A-\lambda I)$ is a polynomial of degree n. In other words, the characteristic equation of an $n\times n$ matrix is always a polynomial of degree n.

Note: In the previous example, eigenvalue 5 is said to have multiplicity 2 because $(\lambda-5)$ occurs two times as a factor of characteristic polynomial.

Ex. The characteristic polynomial of a 6×6 matrix is $\lambda^6-4\lambda^5-12\lambda^4$. Find the eigenvalues and their multiplicities.

$$\lambda^6-4\lambda^5-12\lambda^4=\lambda^4(\lambda^2-4\lambda-12)=\lambda^4(\lambda-6)(\lambda+2)$$

$\lambda=0$ is an eigenvalue (multiplicity 4)

$\lambda=6$ is an eigenvalue (multiplicity 1)

$\lambda=-2$ is an eigenvalue (multiplicity 1)

similarity:

If A & B are $n\times n$ matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP=B$ or equivalently, $A=PBP^{-1}$

changing A to $P^{-1}AP$ is called similarity transformation.

Theorem: If $n\times n$ matrices A & B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicity.

proof: if $B = P^{-1}AP$

Then $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P)$

$= P^{-1}(A - \lambda I)P$

$\Rightarrow \det(B - \lambda I) = \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$

we know $P^{-1}P = I \Rightarrow \det P^{-1} \cdot \det P = \det I = 1$

$\Rightarrow \det(B - \lambda I) = \det(A - \lambda I)$

Note: The converse of this theorem is not correct. e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are not similar although they have the same eigenvalues.

Note: similarity is not the same as row equivalence. Row operations on a matrix often change its eigenvalues.