Diagonalization

Diagonalization is a powerful tool that can be applied to simplify many calculations for matrix A. In general it is easy to do computation w/ Diagonal matrices.

Ex. If $D=\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$, Then find D^2, D^3, and D^k for $k\geq 1$.

$$D^2=\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}5^2 & 0\\ 0 & 3^2\end{bmatrix},\quad D^3=\begin{bmatrix}5^2 & 0\\ 0 & 3^2\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}=\begin{bmatrix}5^3 & 0\\ 0 & 3^3\end{bmatrix}$$

$$\Rightarrow D^k=\begin{bmatrix}5^k & 0\\ 0 & 3^k\end{bmatrix}$$

Note: If a matrix like A can be diagonalized to PDP^{-1} for some invertible matrix P and diagonal D, Then similar computation would be very easy on matrix A as well.

Ex. let $A=\begin{bmatrix}7 & 2\\ -4 & 1\end{bmatrix}$. Find a formula A^k given that $A=PDP^{-1}$ where

$P=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}$ & $D=\begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$.

First, we Find P^{-1}: $\begin{bmatrix}1 & 1 & 1 & 0\\ -1 & -2 & 0 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & -1 & 1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 2 & 1\\ 0 & -1 & 1 & 1\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & -1 & -1\end{bmatrix}\Rightarrow \boxed{P^{-1}=\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}}$$

By simple matrix multiplication:

$$A^2=(PDP^{-1})(PDP^{-1})=(PD)(P^{-1}P)(DP^{-1})=PD^2P^{-1}$$

Similarly $A^3=PD^3P^{-1}$, in general $A^k=PD^kP^{-1}$, Therefore:

$$A^k=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}5^k & 0\\ 0 & 3^k\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}=\begin{bmatrix}2\cdot 5^k-3^k & 5^k-3^k\\ 2\cdot 3^k-2\cdot 5^k & 2\cdot 3^k-5^k\end{bmatrix}$$

Note: In this example, matrix A is said to be diagonalizable if A is similar to a diagonal matrix. That is, if $A = PDP^{-1}$ for some invertible matrix P & some diagonal matrix D, the next Theorem gives a characterization of diagonalizable matrices and tell how to construct a suitable factorization.

Theorem: An $n \times n$ matrix A is diagonalizable if & only if (iff) A has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D, if and only if the columns of P are n linearly independent eigenvectors of A.
In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Ex. Diagonalize the following matrix if possible.

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

In order to solve diagonalization problems, we implement the following steps:

1/ Find the eigenvalues of A

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix} \rightarrow \det(A - \lambda I) = -(1-\lambda)(1+\lambda) - 0 = 0$$

$$= -(1-\lambda^2) = 0 \quad \Rightarrow \lambda = +1, \ \lambda = -1$$

2/ Find the linearly independent eigenvectors

if $\lambda = 1$, $A - I = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$, $6x_1 - 2x_2 = 0 \longrightarrow x_1 = \frac{x_2}{3}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$= 3x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow$ an eigenvector of A

if $\lambda = -1$, $A + I = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x_1 = 0$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

an eigenvector of A

3/ Construct vector P from the vectors in step 2/

$$P=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$ The order of vectors is not important.

4/ Construct the diagonal matrix D from the corresponding eigenvalues.

$$D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Additional step (Sanity check)

It is a good idea to check that P & D that we obtained in the process really work. In order to avoid computing P^{-1}, instead of trying to show $A=PDP^{-1}$, we can just show $AP=PD$

We can do this because
$A=PDP^{-1} \Rightarrow AP=PDP^{-1}P=PDI$
$=PD$

$$P=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A=\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$AP=\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$PD=\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

This means our calculations were correct.

The book example:

Ex. Diagonaliz the following matrix. $A=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, That is, find an invertible matrix P and a diagonal matrix D such That $A = PDP^{-1}$

step 1/ Finding eigenvalues:

$\det(A-\lambda I)=0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 = 0 \Rightarrow -(\lambda-1)(\lambda+2)^2 = 0$

The eigenvalues are $\lambda = 1$ & $\lambda = -2 \rightsquigarrow$ w/ multiplicity 2/

step 2/ Finding Three linearly independent eigenvectors

basis for $\lambda = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, for $\lambda = -2$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

You can check that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set.

step 3/ Construct P from the vectors in step 2/

$$P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

stepe 4/ Construct D from The Corresponding eigenvalues

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Additional step: is $AP = PD$

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

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Ex. Diagonalize the following matrix, if possible. $A=\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

step 1/ The characteristic equation turns out to be the same as the one for previous example.

$$-(\lambda-1)(\lambda+2)^2=0 \quad , \quad \lambda=1 \, , \, \lambda=-2$$

basis for $\lambda=1$, $\vec{v}_1=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

basis for $\lambda=-2$, $v_2=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

We do not have three independent eigenvector for $(n=3)$. Therefore, matrix A is <u>not</u> diagonalizable.

Theorem: An $n \times n$ matrix w/ n distinct eigen values is diagonalizable.

Ex. Determine if the following matrix is diagonalizable.

$$A=\begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Matrix A is a triangular matrix. Therefore, the eigenvalues are the elements on the diagonal of The matrix. This means, $\lambda=5$, $\lambda=0$, and $\lambda=-2$ are three distinct eigen values of The 3×3 matrix. This means The matrix is diagonalizable.

Theorem: let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$ (This means the multiplicity of some of the eigenvalues is larger than one).

Matrix A is diagonalizable iff the sum of the dimension of the eigenspaces equals n, and it happens only if:

i) The characteristic polynomial factors completely into linear factors.

ii) The dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

Ex. Diagonalize the following matrix if possible.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$

$\lambda = 5$ (with multiplicity 2)

$\lambda = -3$ (with multiplicity 2)

$$\lambda = 5, \quad A - \lambda I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ -1 & -2 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ 0 & 2 & -8 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 8 & 16 \\ 0 & 2 & -8 & -8 \end{bmatrix} \longrightarrow \begin{aligned} x_1 &= -8x_3 - 16x_4 \\ 2x_2 &= 8x_3 + 8x_4 \end{aligned}$$

Basic Free

$$\longrightarrow \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8x_3 - 16x_4 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -16x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + x_4 \underbrace{\begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

$\lambda = -3$, using the same procedure we get $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent by the previous Theorem, so $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$ is linearly independent.

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$