Def. of dot product: if $\vec{u}$ & $\vec{v}$ are vectors in $\mathbb{R}^n$, and $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Then $\vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

This number is called the inner product or the dot product and is written as $\vec{u} \cdot \vec{v}$.

Ex. Compute $\vec{u} \cdot \vec{v}$ & $\vec{v} \cdot \vec{u}$ for $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

$\vec{u} \cdot \vec{v} = 2 \times 3 + (-5) \times 2 + (-1) \times (-3) = 6 - 10 + 3 = -1$

$\vec{v} \cdot \vec{u} = 3 \times 2 + 2 \times (-5) + (-3) \times (-1) = 6 - 10 + 3 = -1$

This is not a coincident: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

Properties of dot products (Theorem)

let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then:

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = u \cdot (c\vec{v})$

d) $(\vec{u} \cdot \vec{u}) \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if & only if $\vec{u} = \vec{0}$

These properties can be combined into

$(c_1 \vec{u}_1 + \cdots + c_p \vec{u}_p) \cdot \vec{w} = c_1 (\vec{u}_1 \cdot \vec{w}) + c_2 (\vec{u}_2 \cdot \vec{w}) + \cdots + c_p (\vec{u}_p \cdot \vec{w})$

Definition: The **length** (or **norm**) of $\vec{v}$ is the nonnegative scalar $\|\vec{v}\|$ defined by $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$

and $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

Def. A vector whose length is 1 is called a **unit vector**. If we divide a nonzero vector $\vec{v}$ by its length, that is multiplied by $1/\|\vec{v}\|$, we obtain a unit vector $\vec{u}$.

Ex. let $\vec{v} = (1, -2, 2, 0)$. Find the unit vector $\vec{u}$ in the same direction as $\vec{v}$.

we use $(v_1, v_2, \ldots, v_n)$ notation to save space instead of using $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

First, let's compute the length $\vec{v}$.

$\|\vec{v}\| = \sqrt{1+4+4+0} = 3$ Therefore, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$

we can easily check and see the length of $\vec{u}$ is equal to 1, That is $\|\vec{u}\| = 1$

$$\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0} = \sqrt{\frac{9}{9}} = 1$$

Ex. let W be the subspace of $\mathbb{R}^2$ spanned by $\vec{x} = (2/3, 1)$. find a unit vector z that is a basis for W.

To avoid dealing with fractions, we can define a new vector called $\vec{y} = (2, 3) = 3\vec{x}$. vector $\vec{y}$ is in the same direction as $\vec{x}$. Therefore, we just need to find the unit vector for $\vec{y}$.

$\vec{u} = \dfrac{\vec{y}}{\|\vec{y}\|} = \dfrac{1}{\sqrt{4+9}}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$ (we could multiply $\vec{x}$ with anything else that we wanted to)

for example vector $\begin{bmatrix} -2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}$ is another unit vector that is a basis for W)

Def.: for $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, The **distance between $\vec{u}$ & $\vec{v}$**, written as **dist$(\vec{u},\vec{v})$**, is The length of The vector $\vec{u}-\vec{v}$. That is

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

* In $\mathbb{R}^2$ & $\mathbb{R}^3$ This is The distance of Euclidean distance between two points.

Ex. Compute the distance between the vectors $\vec{u}=(7,1)$ and $\vec{v}=(3,2)$.

$$\vec{u}-\vec{v} = \begin{bmatrix}7\\1\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}4\\-1\end{bmatrix} \quad \text{dist}(u,v) = \sqrt{4^2+(-1)^2} = \sqrt{17}$$

Orthogonal Vectors:

This topic (orthogonality) is based on the concept of perpendicular lines (vectors) in ordinary Euclidean geometry (in $\mathbb{R}^2$ and $\mathbb{R}^3$) has an analogue in $\mathbb{R}^n$.

Def. Two vectors $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\vec{u}\cdot\vec{v}=0$

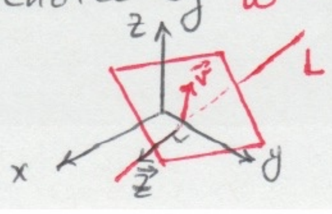
(we can easily prove That in $\mathbb{R}^2$, two lines or two vectors are perpendicular if $\vec{u}\cdot\vec{v}=0$). For The proof we use The vector version of The Pythagorean Theorem that says:

Two vectors $\vec{u}$ & $\vec{v}$ are orthogonal iff $\|\vec{u}+\vec{v}\| = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Def. If a vector $\vec{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, the $\vec{z}$ is said to be orthogonal to W.

Def. The set of all vector $\vec{z}$ That are orthogonal to W is called the **orthogonal complement of W** and is denoted by $W^{\perp}$ and is read W perpendicular or simply W perp.

$$L = W^{\perp}, \quad W = L^{\perp}$$



* **Note:** A vector $\vec{x}$ is in $W^{\perp}$ iff $\vec{x}$ is orthogonal to every vector in a set that span W.

* **Note:** $W^{\perp}$ is a subspace of $\mathbb{R}^n$

Theorem: let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A and the orthogonal complement of the column space of A is the null space of A^T, that is:

$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \& \quad (\text{col } A)^{\perp} = \text{Nul } A^T$$

This not a proof for the Theorem but it gives you an idea about what it means and the general path to prove the Theorem.

if the system $A\vec{x} = \vec{0}$ has nontrivial solution, then $\vec{x}$ is in Nul A (Per definition of Nul A). Remember That $\vec{x}$ is one of Those nontrivial solutions. Therefore,

$$\begin{array}{l} \text{row 1} \\ \text{row 2} \\ \\ \\ \end{array} \underbrace{\begin{bmatrix} \text{———} \\ \text{———} \\ \\ \\ \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vert \\ \vert \\ \vert \\ \\ \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\vec{0}}$$

which means the dot product of every row in A and vector $\vec{x}$ is zero.

This means all The rows of A are orthogonal to The nontrivial solution $\vec{x}$. $\vec{x}$ is also orthogonal to every linear Combination of rows of A. This means the set of all The rows of A and Their linear combinations are the orthogonal Complement of $\vec{x}$ (and hence the Whole Nul A), $(\text{Row } A)^{\perp} = \text{nul } A$

Note: If $\vec{u}$ & $\vec{v}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, Then There is a nice connection b/w their inner product & The angle θ between The two lines:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

when $n > 3$, This formula maybe used to define The angle between Two vectors in $\mathbb{R}^n$. In statistics, for instance, The value of $\cos\theta$ defines what statisticians call a Correlation Coefficient.