Definition: A set of vectors $\{u_1, \ldots, u_p\}$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$

Ex. Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set where:

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

we can consider the following possible pairs:

$\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1(2) + 1(1) = 0$

$\vec{u}_1 \cdot \vec{u}_3 = 3(-1/2) + 1(-2) + 1(7/2) = 0$

$\vec{u}_2 \cdot \vec{u}_3 = (-1)(-1/2) + 2(-2) + 1(7/2) = 0$

Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

Theorem: If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent & hence is a basis for subspace spanned by S.

Definition: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

* **Note:** The reason that we try to deal with orthogonal basis is that it is much easier to find the weights $(c_1, c_2, \ldots, c_n)$ in the linear combination when we deal w/ orthogonal basis.

Orthogonal sets

Theorem: let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. for each $\vec{y}$ in W the weights in the linear combination $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p$ are given by $c_j = \dfrac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$ (for $j = 1, \ldots, p$)

Proof: $\vec{y}\cdot\vec{u}_1 = (c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p)\cdot\vec{u}_1$

$\vec{u}_1\cdot\vec{u}_1$ is not zero ($\vec{u}_1\cdot\vec{u}_1 \neq 0$) but $\vec{u}_1\cdot\vec{u}_j = 0$ for all Values of $j = 2, 3, \ldots, p$

Then $\vec{y}\cdot\vec{u}_1 = c_1(\vec{u}_1\cdot\vec{u}_1) \Rightarrow c_1 = \dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}$, to find c_j for $j = 2, \ldots, p$ we can compute $\vec{y}\cdot\vec{u}_j$ and solve for c_j.

Ex. The set $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Express The vector $\vec{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in S.

$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$.

Solution: $\vec{y} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3$

$$c_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{-12}{6} = -2$$

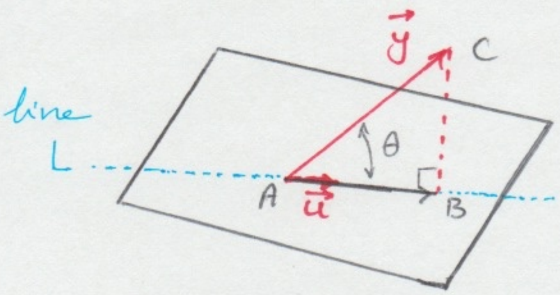
$$c_3 = \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{-33}{-33/2} = -2$$

$$\Rightarrow \vec{y} = \vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3$$

* **Note:** we could solve this problem by forming a system of equations $[\vec{u}_1\ \vec{u}_2\ \vec{u}_3\ \vec{y}]$ and solving it but the process would be more time consuming!

An Orthogonal projection:

Suppose that we have a vector like $\vec{y}$ and we want to find its projection (shadow) along a vector $\vec{u}$.



The angle between $\vec{y}$ & $\vec{u}$ is

$$\cos\theta = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|\cdot\|\vec{y}\|} \quad (1)$$

in the right triangle $\hat{ABC}$: $\cos\theta = \frac{\|\vec{AB}\|}{\|\vec{y}\|}$

$$\Rightarrow \|\vec{AB}\| = \|\vec{y}\|\cos\theta \quad (2)$$

from (1) and (2) $\|\vec{AB}\| = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|}$

This is only the length of vector $\vec{AB}$ (the distance from point A to point B). If we want the vector $\vec{AB}$ itself (and not just its length), we have to multiply the length of $\vec{AB}$, $\|\vec{AB}\|$ by the unit vector along $\vec{u}$, which is $\frac{\vec{u}}{\|\vec{u}\|}$

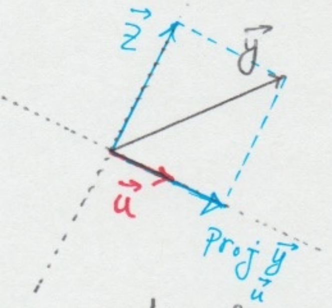
$$\Rightarrow \vec{AB} = \|\vec{AB}\|\cdot\frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|\,\|\vec{u}\|}\,\vec{u} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

This is called orthogonal projection of $\vec{y}$ onto $\vec{u}$ (or along line L that contains $\vec{u}$)

$$\boxed{\operatorname{Proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}}$$

Ex. let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$. Then write y as the sum of two orthogonal vectors, one in Span$\{u\}$ and the other one orthogonal to $\vec{u}$.

$$\begin{cases} \vec{y}\cdot\vec{u} = 40 \\ \vec{u}\cdot\vec{u} = 20 \end{cases} \Rightarrow \text{Proj}_{\vec{u}}\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$



schematic figure

$$\text{Proj}_{\vec{u}}\vec{y} = \frac{40}{20}\,\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

From the figure, we can see $\text{proj}_{\vec{u}}\vec{y} + \vec{z} = \vec{y}$

$$\Rightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \vec{z} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \rightarrow \vec{z} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

orthogonal vector

The sum of these two vectors $\vec{z}$ & $\text{proj}_{\vec{u}}\vec{y}$ should give us $\vec{y}$.

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \vec{y}$$

vector $\vec{z}$ is called component of $\vec{y}$ orthogonal to $\vec{u}$.

Orthonormal sets:

A set $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors. The simplest example of an orthonormal set is the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ for $\mathbb{R}^n$.

e.g. for $\mathbb{R}^3$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex. show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where
$\vec{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$, $\vec{v}_1 \cdot \vec{v}_3 = \frac{-3}{\sqrt{11} \cdot \sqrt{66}} - \frac{4}{\sqrt{11} \cdot \sqrt{66}} + \frac{7}{\sqrt{11} \cdot \sqrt{66}} = 0$

$\vec{v}_2 \cdot \vec{v}_3 = \frac{+1}{\sqrt{6} \cdot \sqrt{66}} - \frac{8}{\sqrt{6} \cdot \sqrt{66}} + \frac{7}{\sqrt{6} \cdot \sqrt{66}} = 0$ Therefore, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.

$\vec{v}_1 \cdot \vec{v}_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = \frac{11}{11} = 1$, $\vec{v}_2 \cdot \vec{v}_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1$

$\vec{v}_3 \cdot \vec{v}_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$ so $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are all unit vectors.

Therefore, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ in an orthonormal set.

Note: Matrices whose columns form an orthonormal set are important for Matrix Computation.

Theorem: An $m \times n$ matrix U has orthonormal columns iff $U^T U = I$

Proof: To simplify the proof, we asume U only has three columns and $U = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$

$$U^T U = \underset{3\times 1}{\begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_3^T \end{bmatrix}} \underset{1\times 3}{[\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]} = \underset{3\times 3}{\begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \vec{u}_1^T\vec{u}_3 \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \vec{u}_2^T\vec{u}_3 \\ \vec{u}_3^T\vec{u}_1 & \vec{u}_3^T\vec{u}_2 & \vec{u}_3^T\vec{u}_3 \end{bmatrix}}$$

Since vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are orthogonal, their inner product is equal to zero. e.g. $\vec{u}_1^T \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_2 = 0$

Also, $\vec{u}_1^T \vec{u}_1 = 1$, $\vec{u}_2^T \vec{u}_2 = 1$, $\vec{u}_3^T \vec{u}_3 = 1$ because vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are all unit vectors.

$$\Rightarrow U^T U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Theorem: let U be an $m \times n$ matrix w/ orthonormal columns, and let $\vec{x}$ and $\vec{y}$ be in $\mathbb{R}^n$. Then:

a. $\|U\vec{x}\| = \|\vec{x}\|$

b. $(U\vec{x})(U\vec{y}) = \vec{x}\cdot\vec{y}$

c. $(U\vec{x})(U\vec{y}) = 0$ iff $\vec{x}\cdot\vec{y} = 0$

This really means $x \mapsto Ux$ transformation preserves the length and orthogonality.

Ex. Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Notice that U has orthonormal columns. verify that $\|U\vec{x}\| = \|\vec{x}\|$

$$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \|U\vec{x}\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|\vec{x}\| = \sqrt{9+2} = \sqrt{11} \quad \Rightarrow \|U\vec{x}\| = \|\vec{x}\|$$