Math 162A
Winter 2021
***Practice* Midterm**
2/1/2021, 12pm-12:50pm
Time Limit: 50 Minutes

Name (Print): ______________________

Teaching Assistant ______________________

This exam contains **??** pages (including this cover page) and **??** problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Upload your answers in pdf format to the Canvas after you finish it.

Run LaTeX again to produce the table

Do not write in the table to the right.

1. (25 points) Let $\alpha = \alpha(s)$ be a plane curve such that all the tangent lines of α pass through a fixed point $p_0 \in \mathbb{R}^2$. Show that α must be a straight line passing through that point p_0.

Solution: The tangent line of $\alpha(t)$ at point t can be written as $\alpha(t) + s\alpha'(t)$. The fact that it must pass a fixed point p_0 means that there is a function $s = s(t)$ such that

$$\alpha(t) + s(t)\alpha'(t) = p_0.$$

We try to solve this ODE: observe that

$$\left(e^{\int \frac{1}{s(t)} dt}(\alpha(t) - p_0)\right)' = 0.$$

Thus there is a constant vector $\mathbf{c}$ such that

$$e^{\int \frac{1}{s(t)} dt}(\alpha(t) - p_0) = \mathbf{c}.$$

Thus

$$\alpha(t) = p_0 + e^{-\int \frac{1}{s(t)} dt}\mathbf{c},$$

which is a straight line.

2. (25 points) Let

$$u(t) = (u_1(t), u_2(t), u_3(t)), \qquad v(t) = (v_1(t), v_2(t), v_3(t))$$

be differential maps from the interval (a,b) into $\mathbb{R}^3$. If the derivatives $u'(t)$ and $v'(t)$ satisfies the conditions

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$

where a, b and c are constants. Show that $u(t) \wedge v(t)$ is a constant vector.

Solution: We have

$$\begin{aligned}(u(t) \wedge v(t))' &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) = 0.\end{aligned}$$

3. (25 points) Let

$$\alpha = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

in $\mathbb{R}^4$. Calculate $d\alpha$ and $\alpha \wedge \alpha$.

Solution:

$$d\alpha = 0,$$

and

$$\begin{aligned} \alpha \wedge \alpha &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\ &= 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

4. (25 points) On $\mathbb{R}^3$, Let K be a real number. Let

$$A = 1 + \frac{K}{4}\sum x_k^2.$$

and let $A_j = \frac{\partial A}{\partial x_j}$. We take

$$\omega_i = \frac{dx_i}{A}.$$

Let

$$\omega_{ij} = \frac{1}{A}(A_i\, dx_j - A_j\, dx_i).$$

Prove that

$$d\omega_j + \omega_i \wedge \omega_{ij} = 0.$$

and

$$d\omega_{ij} + \omega_{is} \wedge \omega_{sj} = K\,\omega_i \wedge \omega_j.$$

Solution: We have

$$d\omega_j = d\frac{1}{A} \wedge dx_j = -\frac{1}{A^2} dA \wedge dx_j = -\frac{A_i}{A^2} dx_i \wedge dx_j.$$

Thus

$$\omega_i \wedge \omega_{ij} = \omega_i \wedge \frac{1}{A}(A_i\, dx_j - A_j\, dx_i) = \frac{A_i}{A}\omega_i \wedge dx_j.$$

Thus

$$d\omega_j + \omega_i \wedge \omega_{ij} = 0.$$

Moreover,

$$d\omega_{ij} = \frac{1}{A}(A_{ik} dx_k \wedge dx_j - A_{jk} dx_k \wedge dx_i) - \frac{1}{A^2} A_k dx_k \wedge (A_i\, dx_j - A_j\, dx_i).$$

We have

$$\frac{1}{A}(A_{ik} dx_k \wedge dx_j - A_{jk} dx_k \wedge dx_i) = \frac{K}{A} dx_i \wedge dx_j,$$

and

$$\omega_{is} \wedge \omega_{sj} = \frac{1}{A^2} A_s dx_s (A_i dx_j - A A_j dx_i) - A_s^2 dx_i \wedge dx_j.$$

Thus

$$d\omega_{ij} + \omega_{is} \wedge \omega_{sj} = \left(\frac{K}{A} - \frac{A_s^2}{A^2}\right) dx_i \wedge dx_j = \frac{K}{A^2} dx_i \wedge dx_j = K\omega_i \wedge \omega_j.$$