Problem 1. (§2.3: 10)

Spherical curves. Let α be a unit-speed curve with $\kappa > 0, \tau \neq 0$.

(a) *If α lies on a sphere of center $\mathbf{c}$ and radius r, show that*

$$\alpha - \mathbf{c} = -\rho N - \rho' \sigma B,$$

where $\rho = 1/\kappa$ and $\sigma = 1/\tau$. Thus $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) *Conversely, if $r^2 = \rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that α lies on a sphere of radius r.*

Solution: Assume that α is on the sphere of radius r. Then we have $\|\alpha - \mathbf{c}\|^2 = r^2$. Taking the derivative, we get

$$(\alpha - \mathbf{c}) \cdot \alpha' = (\alpha - \mathbf{c}) \cdot T = 0.$$

Thus we can write $\alpha - \mathbf{c}$ as a linear combination of N and B:

$$\alpha - \mathbf{c} = aN + bB,$$

where a, b are functions of t. Taking the derivative again and using the Frenet formulas, we get

$$T = \alpha' = a'N + aN' + b'B + bB' = a'N + a(-\kappa T + \tau B) + b'B - b\tau N.$$

From the above equation, we get the following equations

$$1 + a\kappa = 0, \qquad a' - b\tau = 0, \qquad a\tau + b' = 0.$$

We therefore have

$$a = -\rho, \qquad b = -\rho'\sigma.$$

Thus

$$\|\alpha - \mathbf{c}\|^2 = \rho^2 + (\rho'\sigma)^2 = r^2.$$

Conversely, if

$$\alpha - \mathbf{c} = -\rho N - \rho'\sigma B,$$

then

$$\|\alpha - \mathbf{c}\|^2 = \rho^2 + (\rho'\sigma)^2 = r^2.$$

so α is on the sphere of radius r.

■

Problem 2. (§2.3: 11)

Let $\beta, \bar{\beta} : I \to \mathbb{R}^3$ be unit-speed curves with nonvanishing curvature and torsion. If $T = \bar{T}$, then β and $\bar{\beta}$ are parallel (Ex. 10 of Sec. 2). If $B = \bar{B}$, prove that $\bar{\beta}$ is parallel to either β or the curve $s \mapsto -\beta(s)$.

Solution: We consider $(\beta(s) - \bar{\beta}(s))' = T - \bar{T} = 0$. Thus $\beta(s) - \bar{\beta}(s) = \mathbf{c}$ is a constant. Thus β and $\bar{\beta}$ are parallel.

If $B = \bar{B}$, then $B' = \bar{B}'$ and hence

$$-\tau N = -\bar{\tau}\bar{N}.$$

Thus since both N and $\bar{N}$ are unit vectors, we must have

$$N = \pm\bar{N}, \quad \tau = \pm\bar{\tau}.$$

Thus we have

$$-\kappa T + \tau B = N' = \bar{N}' = -\bar{\kappa}\bar{T} + \bar{\tau}\bar{B}.$$

From the above, we conclude that $T = \bar{T}$. Thus β is either parallel to $\bar{\beta}$ or $-\bar{\beta}$. ■

Problem 3. (§2.4: 1)

Express the curvature and torsion of the curve $\alpha(t) = (\cosh t, \sinh t, t)$ in terms of arc length s measured from $t = 0$.

Solution: We have

$$\alpha' = (\sinh t, \cosh t, 1),$$
$$\alpha'' = (\cosh t, \sinh t, 0),$$
$$\alpha''' = (\sinh t, \cosh t, 0).$$

Thus we have

$$\|\alpha'\| = \sqrt{2}\cosh t,$$
$$\alpha' \times \alpha'' = (-\sinh t, \cosh t, -1),$$
$$\|\alpha' \times \alpha''\| = \sqrt{2}\cosh t,$$
$$(\alpha' \times \alpha'') \cdot \alpha''' = 1.$$

We then have

$$\kappa = \tau = \frac{1}{2\cosh^2 t}.$$

In order to find the arc-length reparametrization, we solve the differential equation

$$s'(t) = \|\alpha'(t)\| = \sqrt{2}\cosh t, \quad s(0) = 0.$$

We then have

$$s(t) = \sqrt{2}\sinh t.$$

Therefore

$$\kappa(s) = \tau(s) = \frac{1}{2\cosh^2 t} = \frac{1}{2 + s^2}.$$

■

Problem 4. (§2.4: 4)

Show that the curvature of a regular curve in $\mathbb{R}^3$ *is given by*

$$\kappa^2\nu^4 = \|\alpha''\|^2 - (d\nu/dt)^2.$$

Solution: Since $\nu = ds/dt$, we have

$$\frac{d\nu}{dt} = \frac{d}{dt}\sqrt{\alpha'\cdot\alpha'} = \frac{\alpha'\cdot\alpha''}{\sqrt{\alpha'\cdot\alpha'}} = \frac{\alpha'\cdot\alpha''}{\|\alpha'\|}.$$

We use the identity

$$\|\alpha'\|^2\cdot\|\alpha''\|^2 = (\alpha'\cdot\alpha'')^2 + \|\alpha'\times\alpha''\|^2,$$

to get

$$\|\alpha''\|^2 - (d\nu/dt)^2 = \frac{\|\alpha'\times\alpha''\|^2}{\|\alpha'\|^2}.$$

The formula for the curvature follows from the above result. ∎

Problem 5. (§2.4: 5)

If α *is a curve with constant speed* $c > 0$*, show that*

$$T = \alpha'/c, \qquad N = \alpha''/\|\alpha''\|, \qquad B = \alpha'\times\alpha''/(c\|\alpha''\|),$$
$$\kappa = \frac{\|\alpha''\|}{c^2}, \qquad \tau = \frac{(\alpha'\times\alpha'')\cdot\alpha'''}{c^2\|\alpha''\|^2}.$$

Solution: These formulas follows from a straightforward computation. ∎

Problem 6. (§2.4: 6)

(a). If α *is a cylindrical helix, prove that its unit vector* $\mathbf{u}$ *(Thm. 4.5) is*

$$\mathbf{u} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B$$

and the coefficients here are $\cos\theta$ *and* $\sin\theta$ *(for* θ *as in Def. 4.5).*

(b). Check (a) for the cylindrical helix in Example 4.2 of Chapter 1.

Solution: I think $\mathbf{u}$ should be

$$\mathbf{u} = \pm\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B\right).$$

Since $T\cdot\mathbf{u}$ is a constant, we know that $N\cdot\mathbf{u} = 0$ by taking derivative. Thus we can write

$$\mathbf{u} = a\,T + b\,B,$$

where a is a constant, and b is a function. Taking derivative on both sides, we get

$$0 = a\kappa\,N + b'\,B - b\tau\,N.$$

Thus $b' = 0$ and b must be a constant as well. We also have $a\kappa = b\tau$. since $\mathbf{u}$ is unit, we must have $a^2 + b^2 = 1$. Solve these twp equations, we have

$$a = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \qquad b = \pm\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}.$$

(b). For the curve

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

in Example 4.2 of Chapter 1, we compute

$$\begin{aligned}\alpha' &= (-a\sin t, a\cos t, b),\\ \alpha'' &= (-a\cos t, -a\sin t, 0),\\ \alpha''' &= (a\sin t, -a\cos t, 0).\end{aligned}$$

We then have

$$\alpha' \times \alpha'' = a(b\sin t, -b\cos t, a).$$

Using the formulas in the above problem, we obtain

$$\kappa = \frac{a}{a^2+b^2}, \qquad \tau = \frac{b}{a^2+b^2}.$$

Let $c = \sqrt{a^2+b^2}$. Since

$$T = \frac{1}{c}(-a\sin t, a\cos t, b), \qquad B = \frac{1}{c}(b\sin t, -b\cos t, a),$$

we have

$$\mathbf{u} = \frac{b}{c}T + \frac{a}{c}B = (0,0,1).$$

We thus verify that

$$T\cdot\mathbf{u} = \frac{b}{c}$$

is a constant.

■

Problem 7. (§2.4: 12)

If $\alpha(t) = (x(t), y(t))$ is a regular curve in $\mathbb{R}^2$, show that its plane curvature (Ex. 8 of Sec. 3) is given by

$$\tilde{\kappa} = \frac{\alpha''\cdot J(\alpha')}{\nu^3} = \frac{x'y''-x''y'}{((x')^2+(y')^2)^{3/2}}.$$

Solution: By definition,

$$T = \frac{\alpha'}{\|\alpha'\|}, \qquad \tilde{N} = J(T).$$

We know that

$$T' = \frac{\alpha''}{\|\alpha'\|} - \frac{(\alpha'\cdot\alpha'')\alpha'}{\|\alpha'\|^3}.$$

We thus have

$$\tilde{\kappa} = \nu^{-1}T'\cdot J(T) = \frac{1}{\nu^3}\left(\alpha'' - \frac{(\alpha'\cdot\alpha'')\alpha'}{\|\alpha'\|^2}\right)\cdot J(\alpha') = \frac{\alpha''\cdot J(\alpha')}{\nu^3}.$$

Since $\alpha' = (x', y')$, we have $J(\alpha') = (-y', x')$. Also, we have $\alpha'' = (x'', y'')$. So the second formula follows.

■

Problem 8. (§2.5: 1)

Consider the tangent vector $\mathbf{v} = (1, -1, 2)$ *at the point* $\mathbf{p} = (1, 3, -1)$. *Compute* $\nabla_{\mathbf{v}} W$ *directly from the definition, where*

(a) $W = x^2 U_1 + y U_2$,

(b) $W = x U_1 + x^2 U_2 - z^2 U_3$.

Solution: (a). We have

$$W(\mathbf{p} + t\mathbf{v}) = (1+t)^2 U_1 + (3-t) U_2.$$

Thus

$$\nabla_{\mathbf{v}} W = \frac{d}{dt}\bigg|_{t=0} (1+t)^2 U_1 + (3-t) U_2 = 2\, U_1 - U_2.$$

(b). We have

$$W(\mathbf{p} + t\mathbf{v}) = (1+t)\, U_1 + (1+t)^2\, U_2 - (-1+2t)^2\, U_3.$$

Thus

$$\nabla_{\mathbf{v}} W = \frac{d}{dt}\bigg|_{t=0} (1+t) U_1 + (1+t)^2 U_2 - (-1+2t)^2 U_3 = U_1 + 2\, U_2 + 4\, U_3.$$

■

Problem 9. (§2.5: 3)

If W *is a vector field with constant length* $\|W\|$, *prove that for any vector field* V, *the covariant derivative* $\nabla_V W$ *is everywhere orthogonal to* W.

Solution: Since $W \cdot W = c$, we must have

$$\nabla_V W \cdot W + W \cdot \nabla_V W = 0.$$

Thus $\nabla_V W$ is always orthogonal to W. ■