Chapter 1 Calculus on Euclidean Space

Introduction

- *Calculus on Euclidean space*
- *Tangent Vectors and tangent space*
- *The space of differential forms*
- *The wedge product*
- *The differential operator* d
- *The differential of a mapping*

1.1 Euclidean Space

Definition 1.1

Euclidean 3-space $\mathbb{R}^3$ *is the set of all ordered triples of real numbers. Such a triple* $\mathbf{p} = (p_1, p_2, p_3)$ *is called a* point *of* $\mathbb{R}^3$.

By last chapter, $\mathbb{R}^3$ is a vector space.

Definition 1.2

On $\mathbb{R}^3$*, there are three natural real-valued functions* x, y, z*, defined by*

$$x(\mathbf{p}) = p_1, \quad y(\mathbf{p}) = p_2, \quad z(\mathbf{p}) = p_3.$$

These functions are called natural coordinate functions of $\mathbb{R}^3$.

Remark We shall also use index notation for these functions, writing

$$x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

Definition 1.3

A real-valued function f *on* $\mathbb{R}^3$ *is differentiable (or infinitely differentiable, or smooth, or of class* C^∞*) provided all partial derivatives of* f*, of all orders, exist and are continuous.*

As we know from the previous chapter, the space of smooth functions forms a vector space, that is, let f, g be two smooth functions of $\mathbb{R}^3$ and let $\lambda \in \mathbb{R}$, we have

$$(f+g)(\mathbf{p}) = f(\mathbf{p}) + g(\mathbf{p}), \quad (\lambda f)(\mathbf{p}) = \lambda f(\mathbf{p}).$$

In addition, we have

$$(fg)(\mathbf{p}) = f(\mathbf{p})g(\mathbf{p}).$$

The space of smooth functions, with respect to the three operations: addition, scalar multiplication, and the multiplication forms an *algebra*.

1.2 Tangent Vectors

Definition 1.4

A tangent vector, or a vector $\mathbf{v}_{\mathbf{p}}$ *to* $\mathbb{R}^3$ *consists of two points of* $\mathbb{R}^3$*: its vector part* $\mathbf{v}$ *and its point of application* $\mathbf{p}$. ♣

Definition 1.5

Let $\mathbf{p}$ *be a point of* $\mathbb{R}^3$*. The set* $T_{\mathbf{p}}(\mathbb{R}^3)$ *consisting of all tangent vectors that have* $\mathbf{p}$ *as point of application is called the tangent space of* $\mathbb{R}^3$ *at* $\mathbf{p}$. ♣

Note *Tangent space is a vector space.*

Definition 1.6

A tangent vector field, or a vector field V *on* $\mathbb{R}^3$ *is a function that assigns to each point* $\mathbf{p}$ *of* $\mathbb{R}^3$ *a tangent vector* $V(\mathbf{p})$ *to* $\mathbb{R}^3$ *at* $\mathbf{p}$. ♣

Note *Vector field is one of the most important concepts in differential geometry. By the above definition, a vector field is just a vector valued function. This is because* $\mathbb{R}^3$ *is a flat space, and hence there are global basis under which all tangent spaces can be identified as* $\mathbb{R}^3$*. In general, a vector field defines a different type of "functions" comparing to the traditional one.*

Remark By definition, a vector field doesn't have to be smooth. However, in this course, we always assume it is smooth (or at least of $\mathcal{C}^3$) when regarding it as a vector-valued function.

The domain of a vector field doesn't have to be on the whole $\mathbb{R}^3$: it could be an open set of $\mathbb{R}^3$, or a curve or a surface in $\mathbb{R}^3$. In the latter to cases, we say that the vector field is *along the curve or surface*.

The space of vector fields is obviously a vector space. However, it has finer structure than that. It is a *module* over the algebra of smooth functions.

There are two operations on the space of vector fields: addition and scalar multiplication. Let V, W be two vector fields such that $V = v_i U_i$, $W = w_i U_i$. Let $\lambda \in \mathbb{R}$. Then we can define

$$V + W = \sum_i (v_i + w_i) U_i$$

$$\lambda V = \sum (\lambda v_i) U_i.$$

Moreover, let f be a smooth function of $\mathbb{R}^3$. Then we can define

$$(fV)(\mathbf{p}) = f(\mathbf{p}) V(\mathbf{p})$$

for all $\mathbf{p}$. Of course, such kind of multiplication can be localized to the case when V and f are only defined on a subset of $\mathbb{R}^3$.

Note *The scalar multiplication coincides with the above multiplication by regarding a scalar as a constant function.*

Note *We can also say that the space of vector fields is a module over the ring of smooth functions. But the algebra of smooth functions carries more structure than the ring structure of smooth functions: it has the additional scalar multiple structure. Therefore it is better to say the module over the algebra of smooth functions than that over the ring of smooth functions.*

Definition 1.7

Let U_1, U_2 and U_3 be the vector fields on $\mathbb{R}^3$ such that

$$U_1(\mathbf{p}) = (1,0,0)_{\mathbf{p}}$$
$$U_2(\mathbf{p}) = (0,1,0)_{\mathbf{p}}$$
$$U_3(\mathbf{p}) = (0,0,1)_{\mathbf{p}}$$

for each $\mathbf{p}$ of $\mathbb{R}^3$. We call $\{U_1, U_2, U_3\}$ the natural frame field of $\mathbb{R}^3$. ♣

Remark For fixed point, $\{U_1, U_2, U_3\}$ provides the standard basis of $\mathbb{R}^3$, usually expressed as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The following result is a generalization of what we have learned in linear algebra.

Lemma 1.1

If V is a vector field of $\mathbb{R}^3$, then there are three uniquely determined real-valued functions v_1, v_2, v_3 on $\mathbb{R}^3$ such that

$$V = v_1U_1 + v_2U_2 + v_3U_3.$$

These three functions are called Euclidean coordinate functions of V. ♡

Proof. For fixed $\mathbf{p} \in \mathbb{R}^3$, $V(\mathbf{p})$ defines a *vector* in $\mathbb{R}^3$, therefore there is unique numbers $v_1(\mathbf{p})$, $v_2(\mathbf{p})$, and $v_3(\mathbf{p})$ such that

$$V(\mathbf{p}) = v_1(\mathbf{p})U_1(\mathbf{p}) + v_2(\mathbf{p})U_2(\mathbf{p}) + v_3(\mathbf{p})U_3(\mathbf{p}).$$

Thus

$$V = v_iU_i$$

by *definition*. ∎

1.3 Directional Derivatives

Definition 1.8

Let f be a differentiable real-valued function on $\mathbb{R}^3$, and let $v_{\mathbf{p}} \in T_{\mathbf{p}}(\mathbb{R}^3)$ be a tangent vector to $\mathbb{R}^3$. Then the number

$$v_{\mathbf{p}}[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))|_{t=0}$$

is called the derivative of f with respect to $v_{\mathbf{p}}$ ♣

Note *It is called the directional derivative because* $\mathbf{p} + t\mathbf{v}$ *for non-negative real number* t *represents a ray starting from* $\mathbf{p}$ *in the direction* $\mathbf{v}$. *We have encountered directional derivative in Calculus. Here the emphasis is that "vector" (which is an algebraic concept) can be identified as a "derivative" (which is a calculus concept).*

Lemma 1.2

If $v_{\mathbf{p}} = (v_1, v_2, v_3)$ *is a tangent vector to* $\mathbb{R}^3$, *then*

$$v_{\mathbf{p}}[f] = \sum v_i \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

Proof. Let $\mathbf{p} = (p_1, p_2, p_3)$; then

$$\mathbf{p} + t\mathbf{v} = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3).$$

We then

$$v_{\mathbf{p}}[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))|_{t=0} = \sum_i \frac{\partial f}{\partial x_i}(\mathbf{p})v_i.$$

■

Example 1.1 Let $f(x, y, z) = x^2yz$. Let $\mathbf{p} = (1, 1, 0)$ and $\mathbf{v} = (1, 0, -3)$. Then

$$\frac{\partial f}{\partial x} = 2xyz, \qquad \frac{\partial f}{\partial y} = x^2z, \qquad \frac{\partial f}{\partial z} = x^2y.$$

Thus

$$\frac{\partial f}{\partial x}(\mathbf{p}) = 0, \qquad \frac{\partial f}{\partial y}(\mathbf{p}) = 0, \qquad \frac{\partial f}{\partial z}(\mathbf{p}) = 1.$$

Thenfore

$$v_{\mathbf{p}}[f] = 0 + 0 + 1 \cdot (-3) = -3.$$

Theorem 1.1

Let f *and* g *be functions on* $\mathbb{R}^3$, $v_{\mathbf{p}}$ *and* $w_{\mathbf{p}}$ *tangent vectors,* a *and* b *numbers. Then*

- $(av_{\mathbf{p}} + bw_{\mathbf{p}})[f] = av_{\mathbf{p}}[f] + bw_{\mathbf{p}}[f]$;
- $v_{\mathbf{p}}(af + bg) = av_{\mathbf{p}}[f] + bv_{\mathbf{p}}[g]$;
- $v_{\mathbf{p}}[fg] = v_{\mathbf{p}}[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot v_{\mathbf{p}}[g]$.

Proof. Only the 3rd equation is new, which can be proved using Lemma 1.2: We have

$$v_{\mathbf{p}}[fg] = v_i \frac{\partial (fg)}{\partial x_i}(\mathbf{p}) = v_i f(\mathbf{p}) \frac{\partial g}{\partial x_i}(\mathbf{p}) + v_i g(\mathbf{p}) \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

By definition, we have

$$v_{\mathbf{p}}[fg] = v_{\mathbf{p}}[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot v_{\mathbf{p}}[g].$$

■

1.4 Curves in $\mathbb{R}^3$

One of the fundamental questions in curve theory is: how to define a curve? In Euclidean geometry, only two kinds of curves are studied: straight line and circle. In analytic geometry, we study parabola, ellipse, and hyperbola. These curves have quite explicit geometric meanings. For example, an ellipse is the the set of all points in a plane such that the sum of the distances from two fixed points (foci) is constant. If we want to study more general curves, we should not expect them have clear geometric meanings.

In differential geometry, we define a curve in $\mathbb{R}^3$ by a function

$$\alpha : I \to \mathbb{R}^3, \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)),$$

where I is an open interval. In order to use calculus, we usually assume that such a function is smooth.

Definition 1.9

A curve in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$ from an open interval into $\mathbb{R}^3$. ♣

We shall give a couple of examples of curves.

Example 1.2 (Straight Line) A *straight line* can be expressed best using the vector notations. Let $\mathbf{p}$, $\mathbf{q}$ be two vectors and let $\mathbf{q} \neq 0$. Then we can use

$$\alpha(t) = \mathbf{p} + t\mathbf{q}$$

to represent a curve with direction $\mathbf{q}$.

Example 1.3 (Helix) The parameter equations for a circle (in $\mathbb{R}^3$) can be expressed by

$$t \mapsto (a\cos t, a\sin t, 0).$$

If we allow this curve to rise , then we obtain a *helix* $\alpha : \mathbb{R} \to \mathbb{R}^3$, given by the formula

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

where $a > 0$ and $b \neq 0$.

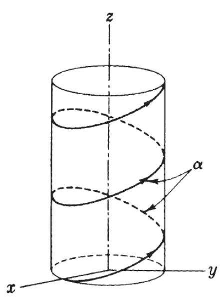

FIG. 1.7

Definition 1.10

Let $\alpha : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each number $t \in I$, the velocity vector of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right)_{\alpha(t)}$$

at the point $\alpha(t)$ in $\mathbb{R}^3$.

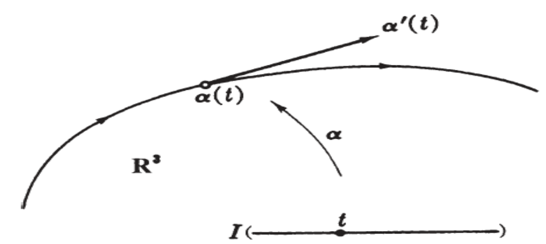

FIG. 1.9

;

Example 1.4 For the helix,

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

the velocity vector is

$$\alpha'(t) = (-a\sin t, a\cos t, b)_{\alpha(t)}.$$

Definition 1.11

Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $h : J \to I$ is a differentiable function on an open interval J, then the composition function

$$\beta = \alpha(h) : J \to \mathbb{R}^3$$

is a curve called a reparametrization of α by h.

Note *The above definition is a key concept. See the next lemma.*

Lemma 1.3

If β is the reparametrization of α by h, then

$$\beta'(s) = \frac{df}{ds} \cdot \alpha'(h(s)).$$

Proof. This is a straightforward application of the chain rule. ∎

Lemma 1.4

Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

We end up this section by a discussion of *dual space*.

Dual Space

Definition 1.12

Given any vector space V, the dual space V^* is defined as the set of all linear transformations $\varphi : V \to \mathbb{R}$. The dual space is a vector space by the following definition of addition and scalar multiplication.

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a(\varphi(x))$$

for all $\varphi, \psi \in V^*$ and $a \in \mathbb{R}$, $x \in V$. Elements of V^* is called a covector, or linear functional, or a one-form. ♣

Example 1.5 On $\mathbb{R}^n$, any linear function

$$\ell(\mathbf{x}) = c_1x_1 + \cdots + c_nx_n,$$

where $\mathbf{x} = (x_1, \cdots, x_n)$ and $c_1, \cdots, c_n$ being real numbers, is a liner functional.

Example 1.6 Let $\mathcal{C}([0, 1])$ be the vector space of continuous functions over $[0, 1]$. Then

$$f \mapsto \int_0^1 f(x)dx$$

defines a linear functional.

Example 1.7 Let $\mathbf{p} \in \mathbb{R}^3$. Let $\mathbf{v_p}$ be a vector on $T_{\mathbf{p}}(\mathbb{R}^3)$. Then the directional derivative

$$f \mapsto \mathbf{v_p}[f]$$

is a linear functional on the vector space of differentiable functions.

External Link. *Here is a good video of the dual space. The first 8 minutes is useful, and the last part is beyond the scope of this course.*

Theorem 1.2. (The Riesz Representation Theorem)

Let V be a finite dimensional vector space and let $\langle\ ,\ \rangle$ be an inner product of V. Let $\mathbf{x} \in V$. Then it defines a linear functional $\ell_{\mathbf{x}}$ such that $\ell_{\mathbf{x}}(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for any $\mathbf{y} \in V$; conversely, let ℓ be a linear functional, then there is a unique $\mathbf{x} \in V$ such that $\ell(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for any $\mathbf{y} \in V$. ♡

Remark In order words, every linear functional can be *represented*, through a fixed inner product, as an element of the vector space.

Remark We elaborate the Riesz Representation Theorem in the context of the vector space $\mathbb{R}^n$ with the dot product. Let $\mathbf{x} \in \mathbb{R}^n$, and let ℓ be a linear functional. By the above theorem, there is a vector $\mathbf{c}$ such that

$$\ell(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = c_1x_1 + \cdots + c_nx_n.$$

In this way, the dual space of $\mathbb{R}^n$ can be identified to $\mathbb{R}^n$.

Note *There is an infinite dimensional version of the Riesz Representation Theorem on normed* *vector space, but the linear functional in question needs to be replaced by bounded linear functional.*

External Link. *The linear algebra over infinite dimensional vector spaces is called Functional Analysis.*

1.5 1-forms

Definition 1.13

A 1-form ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is,

$$\phi(a\mathbf{v} + b\mathbf{w}) = a\phi(\mathbf{v}) + b\phi(\mathbf{w})$$

for any number a, b and tangent vectors $\mathbf{v}, \mathbf{w}$ at the same point of $\mathbb{R}^3$.[a]

[a] Recall in Definition 1.4, a vector on $\mathbb{R}^3$ is a pair $(\mathbf{p}, \mathbf{v})$, where $\mathbf{p} \in \mathbb{R}^3$ and $\mathbf{v}$ is the vector part.

♣

As before, the space of 1-forms is a module over the algebra of smooth functions. Let φ, ψ be two 1-forms; let $\mathbf{v}$ be a vector on $\mathbb{R}^3$; let $\lambda \in \mathbb{R}$. Then we can define

$$(\varphi + \psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v}); \qquad (\lambda\varphi)(\mathbf{v}) = \lambda\varphi(\mathbf{v}).$$

Note $\varphi(\mathbf{v})$ is a smooth function on $\mathbb{R}^3$. Let f be a smooth function. Let $\mathbf{v_p}$ be the vector part of $\mathbf{v}$.

$$(f\varphi)(\mathbf{v_p}) = f(\mathbf{p})\varphi(\mathbf{v_p}).$$

In fact, there is a natural way to extend a 1-from as a function over vector fields. A 1-form is a linear functional in two ways: first, it is a linear functional over the vector space of vector fields, that is, if φ is a 1-form, for any vector field $\mathbf{v}$, $(\varphi(\mathbf{v}))(\mathbf{p}) = \varphi(\mathbf{v_p})$ is a smooth function; second, for any fixed point $\mathbf{p}$, φ is a linear functional over $T_{\mathbf{p}}(\mathbb{R}^3)$.

Definition 1.14

If f is a differentiable function on $\mathbb{R}^3$. Then df is a 1-form defined by

$$df(\mathbf{v_p}) = \mathbf{v_p}[f].$$

♣

Example 1.8 1-forms on $\mathbb{R}^3$: by the above definition, we can define 1-froms dx_1, dx_2, dx_3. Let

$$\mathbf{v_p} = v_i U_i.$$

Then by definition,

$$dx_i[\mathbf{v_p}] = \mathbf{v_p}[x_i] = v_i.$$

Let's consider the 1-form

$$\psi = f_i dx_i,$$

where f_i are functions. Then

$$\psi[\mathbf{v_p}] = f_i dx_i[\mathbf{v_p}] = f_i(\mathbf{p})v_i.$$

Definition 1.15. Dual Basis

Let V be a vector space and let $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ be a basis of V. $(\mathbf{f}_1, \cdots, \mathbf{f}_n)$ is called the dual basis of $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$, if $\mathbf{f}_i \in V^*$, and if

$$\mathbf{f}_i(\mathbf{e}_j) = \delta_{ij}.$$

♣

Theorem 1.3. Dual Basis Theorem

Using the above notations, then (dx_1, dx_2, dx_3) is the dual basis of (U_1, U_2, U_3).

♡

Proof. We have

$$dx_i(U_j) = U_j[x_i] = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

■

Corollary 1.1

If f is a differentiable function on $\mathbb{R}^3$, then

$$df = \frac{\partial f}{\partial x_i} dx_i = f_i dx_i.$$

♡

Note *As we have seen, we use f_i to represent $\frac{\partial f}{\partial x_i}$. In differential geometry, this would greatly simplify complicated computations. In general, whether f_i is $\frac{\partial f}{\partial x_i}$ or an arbitrary function depends on the context.*

Proof. By definition, $df[\mathbf{v_p}] = \mathbf{v}_p[f]$. But $\mathbf{v}_p[f] = v_i f_i(\mathbf{p}) = (f_i dx_i)[\mathbf{v_p}]$.

■

From the definition of df, we observed that we can regard d as an operator, that would send a function f to a 1-form df. Such an operator is called a *differential operator*, which plays one of the center role in differential geometry.

Lemma 1.5

Let fg be the product of differentiable functions f and g on $\mathbb{R}^3$. Then

$$d(fg) = gdf + fdg.$$

♡

Proof. We have

$$d(fg) = (fg)_i dx_i = (gf_i + fg_i)dx_i = gf_i dx_i + fg_i dx_i = g\,df + f dg.$$

■

Lemma 1.6

Let f be a function on $\mathbb{R}^3$ and let $h : \mathbb{R} \to \mathbb{R}$ be a function of single variable. Then

$$d(h(f)) = h'(f)df.$$

♡

Proof.

$$d(h(f)) = (h(f))_i dx_i = h'(f) f_i dx_i = h'(f)df.$$

■

Example 1.9 Let f be the function

$$f(x, y) = (x^2 - 1)y + (y^2 + 2)z.$$

Then

$$df = 2xydx + (x^2 + 2yz - 1)dy + (y^2 + 2)dz.$$

As a result, we have

$$df[\mathbf{v}] = 2xyv_1 + (x^2 + 2yz - 1)v_2 + (y^2 + 2)v_3.$$

Thus

$$df[\mathbf{v_p}] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_3^2 + 2)v_3.$$

We also have

$$\mathbf{v_p}[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_3^2 + 2)v_3.$$

This verifies $df[\mathbf{v_p}] = \mathbf{v_p}[f]$.

1.6 Differential Forms

The space of differential 1-forms is a vector space, or more precisely, it is a module over the algebra of smooth functions. To get more information from tangent spaces where the space of differential 1-forms are dual spaces of them at each point, we shall define *multiplication* of differential 1-forms. Since all differential 1-forms are generated by dx_1, dx_2, dx_3, we just need to define their multiplications.

What is $dx_i dx_j$, or we called the *wedge* product $dx_i \wedge dx_j$ of them? We don't know at this moment. But we shall assume that

$$dx_i dx_j = -dx_j dx_i$$

for $1 \leq i, j, \leq 3$. Obviously, this would create a new kind of algebra. For multiplication of real numbers, we have commutativity, which means, for any two real numbers a, b, we have $ab = ba$. On the other hand, for two $n \times n$ matrics A, B, in general, we have $AB \neq BA$. The property for the multiplication of 1-forms are different from both of the above two. It is called *skew commutativity*. The algebra defined by the skew commutativity leads to the so-called *exterior* *algebra*.

A first observation on the definition of the wedge product reveals that, since $dx_i \wedge dx_i = -dx_i \wedge dx_i$, we must have $dx_i \wedge dx_i = 0$. A quick counting shows that the only non-zero independent products would be $dx_1 dx_2, dx_1 dx_3$ and $dx_2 dx_3$.

In general, we can define the whole system of p-forms. we have already encountered 0-forms, which are smooth functions, and 1-forms. Taking multiplication of dx_i with dx_j, we can define the space of two forms to be generated by $dx_1 dx_2, dx_1 dx_3$ and $dx_2 dx_3$ over smooth functions, that is, all two forms can be expressed by

$$f dx_1 dx_2 + g dx_1 dx_3 + h dx_2 dx_3,$$

where f, g, h are functions.

We can define the 3-forms in an obvious way: all three forms have the expressions

$$f dx_1 dx_2 dx_3,$$

where f is a function.

In the high dimensional case, we can define the p-forms for $p > 3$. However, on $\mathbb{R}^3$, all higher differential forms would be zero: consider, for example, a 4-form $dx_i dx_j dx_k dx_l$. Since the space is of 3 dimensional, at least two of the indices must be the same. By skew commutativity, all 4-forms must be zero.

Example 1.10 Compute the Wedge products

(1). Let

$$\phi = x dx - y dy, \qquad \psi = z dx + x dz.$$

Then

$$\begin{aligned}
\phi \wedge \psi &= (x dx - y dy) \wedge (z dx + x dz) \\
&= xz dx dx + x^2 dx dz - yz dy dx - xy dy dz \\
&= yz dx dy + x^2 dx dz - xy dy dz.
\end{aligned}$$

(2). Let $\theta = z dy$. Then

$$\theta \wedge \phi \wedge \psi = -x^2 z\, dx dy dz.$$

(3). Let $\eta = y dx dz + x dy dz$. Then

$$\begin{aligned}
\phi \wedge \eta &= (x dx - y dy) \wedge (y dx dz + x dy dz) \\
&= (x^2 + y^2)\, dx dy dz.
\end{aligned}$$

Note *It should be clear from these examples that the wedge product of a p-form and a q-form is a $(p+q)$-form. Thus such a product is automatically zero whenever $p+q > 3$.*

Lemma 1.7

Let ϕ, ψ be 1-forms. Then

$$\phi \wedge \psi = -\psi \wedge \phi.$$

Proof. Let $\phi = f_i dx_i, \psi = g_i dx_i$. Then

$$\phi \wedge \psi = f_i dx_i g_j dx_j = f_i g_j dx_i dx_j = -f_i g_j dx_j dx_i = -\psi \wedge \phi.$$

■

Remark The space of any p-forms forms a module over smooth functions[1]. However, given that a p-form wedge a q-form to be a $(p+q)$-form, we can take the *direct sum* of the modules of all p-forms. Obviously, this would give us a module over functions where the wedge product is well defined.

In what follows we will define arguably the most important concept in differential geometry.

Definition 1.16

If $\phi = f_i dx_i$ is a 1-form on $\mathbb{R}^3$. The exterior derivative, or differential, of ϕ is the 2-form

$$d\phi = df_i \wedge dx_i.$$

♣

Example 1.11 If

$$\phi = f_i dx_i = f_1 dx_1 + f_2 dx_2 + f_3 dx_3,$$

then we have

$$d\phi = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 dx_3.$$

Thus if we identify ϕ to a vector-valued function $E = (f, f_2, f_3)$, then $d\phi$ can be identified as $\mathbf{curl}(E)$. In this sense, $d\phi$ generalize the **curl** operator.

Theorem 1.4

Let f, g be functions and ϕ, ψ be 1-forms. Then

1. $d(fg) = df\, g + f\, dg$;
2. $d(f\phi) = df \wedge \phi + f\, d\phi$;
3. $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$[a].

[a] This is more a definition than a property of the differential operator.

♡

Proof. Property (1) is just the product rule We proved in Lemma 1.5. To prove (2), we let $\phi = f_i dx_i$. Then

$$d(f\phi) = d(ff_i) \wedge dx_i = df \wedge f_i dx_i + f df_i \wedge dx_i = df \wedge \phi + f\, d\phi.$$

Property (3) is, straightly speaking, a definition rather than a property, since we have never defined the differential of 2-forms before. Nevertheless, let's work on it. First,

$$d(\phi \wedge \psi) = d(f_i g_j dx_i dx_j).$$

As in the case of 1-forms, we define

$$d(f_i g_j dx_i dx_j) = d(f_i g_j) \wedge dx_i \wedge x_j.$$

[1] For any p, even if $p > 3$ or $p < 0$, where we defined the module to be 0.

We then have

$$\begin{aligned} d(\phi \wedge \psi) &= df_i g_j \wedge dx_i \wedge dx_j + f_i dg_j \wedge dx_i \wedge dx_j \\ &= df_i \wedge dx_i \wedge g_j dx_j - f_i \wedge dx_i \wedge dg_j \wedge dx_j \\ &= d\phi \wedge \psi - \phi \wedge d\psi. \end{aligned}$$

■

Example 1.12 Let

$$\phi = f_1 dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2$$

be a 2-form. Then

$$d\phi = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 dx_2 dx_3.$$

Thus if we identify $E = (f_1, f_2, f_3)$. Then $d\phi$ can be identify to $\mathbf{div}(E)$.

Example 1.13 Let f be a function, then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

can be identified as ∇f.

Example 1.14 If we identify $\phi = f_i dx_i$ to $\mathbf{u}$ and $\psi = g_i dx_i$ to $\mathbf{v}$, then $\phi \wedge \psi$ can be identified to $\mathbf{u} \times \mathbf{v}$.

✍ **Exercise 1.1** Can you use exterior algebra to define the dot product?

1.7 Mappings

In this section we discuss functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $n = 3$ and $m = 1$, this is just a function on $\mathbb{R}^3$. In the other extreme, if $n = 1$ and $m = 3$, then this is a single variable $\mathbb{R}^3$-valued function, and by the previous sections, they can be used to represent curves in $\mathbb{R}^3$. All of these functions have been studied in Calculus, but in this section, we shall study them using the idea of *linearization*.

Recall that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 1.17

Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$, let $f_1, \cdots, f_m$ denote the real-valued function on $\mathbb{R}^n$ such that

$$F(\mathbf{p}) = (f_1(\mathbf{p}), f_2(\mathbf{p}), \cdots, f_m(\mathbf{p}))$$

for all points $\mathbf{p} \in \mathbb{R}^n$. These functions are called the Euclidean coordinate functions of F, and we can write $F = (f_1, \cdots, f_m)$.

The functions F is differentiable provided its coordinate functions are differentiable in the usual sense. A differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. ♣

Definition 1.18

If $\alpha : I \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then the composition function $\beta = F(\alpha) : I \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ called the image of α under F. ♣

Note *Let $\mathscr{B}$ be the set of all curves in $\mathbb{R}^n$ and let $\mathscr{C}$ be the set of all curves in $\mathbb{R}^m$. Then a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ induces a map $\mathscr{B} \to \mathscr{C}$.*

Definition 1.19

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\mathbf{v}$[a] is tangent vector to $\mathbb{R}^n$ at $\mathbf{p}$. Let $F_*(\mathbf{v})$ be the initial velocity of the curve $t \mapsto F(\mathbf{p} + t\mathbf{v})$. The resulting function F_* sends a tangent vectors to $\mathbb{R}^n$ to tangent vectors to $\mathbb{R}^m$, and is called the tangent map of F.

[a] If $\mathbf{v} = 0$, the straight line $\mathbf{p} + t\mathbf{v}$ is degenerated to a point $\mathbf{p}$. But the definition is still valid. ♣

Proposition 1.1

Let $F = (f_1, \cdots, f_m)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $\mathbf{v}$ is a tangent vector to R^n at $\mathbf{p}$, then

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], \cdots, \mathbf{v}[f_m])$$

at $F(\mathbf{p})$. ♠

Proof. We take $m = 3$ for simplicity. By definition, the curve $t \mapsto F(\mathbf{p} + t\mathbf{v})$ can be written as

$$\beta(t) = F(\mathbf{p} + t\mathbf{v}) = (f_1(\mathbf{p} + t\mathbf{v}), f_2(\mathbf{p} + t\mathbf{v}), f_3(\mathbf{p} + t\mathbf{v})).$$

By definition, we have $F_*(\mathbf{v}) = \beta'(0)$. To get $\beta'(0)$, we take the derivatives, at $t = 0$, of the coordinate functions of β. But

$$\frac{d}{dt}(f_i(\mathbf{p} + t\mathbf{v}))|_{t=0} = \mathbf{v}[f_i].$$

Thus

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], \mathbf{v}[f_2], \mathbf{v}[f_3])|_{\beta(0)},$$

where $\beta(0) = F(\mathbf{p})$. ■

Let $\mathbf{p} \in \mathbb{R}^n$. Then we have the linear transformation

$$F_{*\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to T_{F(\mathbf{p})}(\mathbb{R}^m)$$

called the tangent map of F at $\mathbf{p}$.

Corollary 1.2

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then at each point $\mathbf{p}$ of $\mathbb{R}^n$, the tangent map $F_{*\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to T_{F(\mathbf{p})}(\mathbb{R}^m)$ is a linear transformation. ♡

Note *For any nonlinear function F, we can define a semi-linear function $F_{*\mathbf{p}}$, where for fixed $\mathbf{p}$, the function is a linear transformation. But the function $F_* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, $(\mathbf{p}, \mathbf{v}) \mapsto F_{*\mathbf{p}}(\mathbf{v})$ is nonlinear with respect to $\mathbf{p}$.*

Note *Let $f(t)$ be a function of single variable. Then $f_* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(t, s) \mapsto sf'(t)$ is the tangent map of f. Such a tangent map can be identified to the derivative $f'(t)$.*

Corollary 1.3

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$. ♡

Corollary 1.4

If $F = (f_1, \cdots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$F_*(U_j(\mathbf{p})) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\mathbf{p})\bar{U}_i(F(\mathbf{p})),$$

where $\{\bar{U}_i\}$, for $i = 1, \cdots, m$ are natural frame fields of $\mathbb{R}^m$. ♡

Definition 1.20

The matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{p})\right)_{1\le i\le m, 1\le j\le n}$$

is called the Jacobian matrix of F at $\mathbf{p}$. ♣

In terms of matrix notations, we have

$$F_*[U_1(\mathbf{p}), \cdots, U_n(\mathbf{p})] = [\bar{U}_1(F(\mathbf{p})), \cdots, \bar{U}_m(F(\mathbf{p}))] \cdot J.$$

Definition 1.21

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is regular provided that at every point $\mathbf{p}$ of $\mathbb{R}^n$ the tangent map $F_{*\mathbf{p}}$ is one-to-one. ♣

Remark By linear algebra, the following are equivalent

(1) $F_{*\mathbf{p}}$ is one-to-one.

(2) $F_{*\mathbf{p}}(\mathbf{v}_\mathbf{p}) = 0$ implies $\mathbf{v}_\mathbf{p} = 0$.

(3) The Jacobian matrix of F at $\mathbf{p}$ has rank n, the dimension of the domain $\mathbb{R}^n$ of F.

Remark If $m = n$, then we know that, by the Invertible Matrix Theorem, that $F_{*\mathbf{p}}$ is one-to-one if and only if it is onto.

Definition 1.22

Let $\mathscr{U}, \mathscr{V}$ *be two open sets of* $\mathbb{R}^n$. *We say that* $\mathscr{U}$ *and* $\mathscr{V}$ *are* *diffeomorphic*, *if there is a differentiable map* $F : \mathscr{U} \to \mathscr{V}$ *which is one-to-one and onto. Moreover, the inverse mapping:* $F^{-1} : \mathscr{U} \to \mathscr{V}$ *is also differentiable. We also say that* F *is a* *diffeomorphism* *of* $\mathscr{U}$ *to* $\mathscr{V}$.

♣

Theorem 1.5. (Inverse Function Theorem)

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ *be a mapping between Euclidean spaces of the same dimension. If* $F_{*\mathbf{p}}$ *is one-to-one at a point* $\mathbf{p}$, *there is an open set* $\mathscr{U}$ *containing* $\mathbf{p}$ *such that* F *restricted to* $\mathscr{U}$ *is a diffeomorphism of* $\mathscr{U}$ *onto an open set* $\mathscr{V}$.

♡