Differential Geometry @ UCI

Math 162AB

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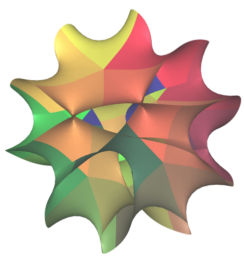

Lecture Notes based on **Barrett O'Neill**'s book *Elementary Differential Geometry*.

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0.1 Vector Spaces

Definition 0.1. Vector Space

A vector space V is a nonempty set with two binary operations "$+$" and scalar multiplication "$\cdot$" satisfying the following eight axioms: let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbb{R}$, we have

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
- $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ *for a vector called* $\mathbf{0}$;
- $(rs) \cdot \mathbf{u} = r \cdot (s \cdot \mathbf{u})$;
- $(r + s) \cdot \mathbf{u} = r \cdot \mathbf{u} + s \cdot \mathbf{u}$;
- $r \cdot (\mathbf{u} + \mathbf{v}) = r \cdot \mathbf{u} + r \cdot \mathbf{v}$;
- $0 \cdot \mathbf{u} = \mathbf{0}$;
- $1 \cdot \mathbf{u} = \mathbf{u}$.

♣

As is well-known, the set V satisfying the first three axioms form an *Abel semi-group*. The existence of the inverse of a vector $\mathbf{u}$ can be verified by using the scalar multiplication. Let $\mathbf{u}$ be a vector, we claim that $(-1) \cdot \mathbf{u}$ is the inverse of $\mathbf{u}$ because

$$\mathbf{u} + (-1) \cdot \mathbf{u} = (1 + (-1)) \cdot \mathbf{u} = \mathbf{0}.$$

Similarly, we can define the subtraction by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1) \cdot \mathbf{v}.$$

Note *We sometimes omit the $\cdot$ and write, for example, $r\mathbf{u}$ for $r \cdot \mathbf{u}$.*

External Link. *Here is the video explanation of vector space (linear indepedence).* *Useful!*

External Link. *Here is the Math 162A pre-requisite videos.*

In linear algebra, we restrict ourselves to finite dimensional vector space. But, a lot of results in finite dimensional case can be extended to infinite dimensional case as well as *abstract* vector space cases.

In the following, we give some examples of vector spaces.

Example 0.1 $\mathbb{R}^n$, the n-dimensional *Euclidean space*, is the set of all n-vectors

$$\mathbb{R}^n = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_i \in \mathbb{R} \right\}.$$

Example 0.2 The set of all $m \times n$ matrices for an (mn)-dimensional vector space.

Example 0.3 The space of polynomials of degree no more than n, where n is a nonnegative integer, is a vector space.

A single-variable polynomial of degree no more than n can be expressed as

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n.$$

Let

$$q(t) = b_0 + b_1 t + \cdots + b_n t^n$$

be another polynomial. Define the addition to be

$$(p+q)(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n,$$

and the scalar multipliction to be: let $\lambda \in \mathbb{R}$

$$(\lambda p)(t) = (\lambda a_0) + (\lambda a_1)t + \cdots + (\lambda a_n)t^n.$$

With respect to the addition and scalar multiplication, the space is a vector space.

In the above two examples, the dimensions of the vector spaces are *finite*. Let's show some examples of infinite dimensional vector spaces.

Note *The set of all polynomials of degree equal to n is not a vector space.*

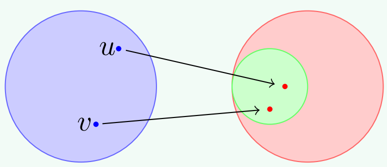
Example 0.4 Moreover, the space of all real-valued functions on a set is a (infinite dimensional) vector space.

Let's recall the definition of a function.

Definition 0.2

A function $f : X \to Y$ is a triple (f, X, Y), where X, Y are sets, and f is an assignment, or a rule, that for any element x in X, there is a unique $y = f(x)$ in Y attached to it. X is called the domain of f, and Y is called the codomain of f. The range is the subset of the codomain Y consists of all $f(x)$ when X is running through X. The assignment sometimes is written as $x \mapsto f(x)$. Thus a complete description of a function can be given as

$$f : X \to Y, \quad x \mapsto f(x).$$



As above, we can define the addition and scalar multiplication as

$$(f+g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda f(x).$$

Remark The vector space we study in differential geometry are the "abstract" vector space, which is on the contrary to the vector spaces we studied in Math 3A.

The following concepts are defined in abstract vector spaces similar to those in $\mathbb{R}^n$.

1. linear combination, span;
2. linear dependence and independence;
3. basis and dimension.

Remark In infinite dimensional space, a basis is defined by a set of vectors

$$\mathfrak{I} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \cdots\}$$

satisfying the following

1. any finite subset of $\mathfrak{I}$ is linearly independent;
2. any element can be expressed as a (finite) linear combination of element in $\mathfrak{I}$.

Let $(V, +, \cdot)$ be a vector space. We can endow geometric structure onto it by defining the concept of inner product.

Remark Let

$$\mathfrak{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$$

be a finite set. Let

$$V = \operatorname{Span} \mathfrak{S} = \operatorname{Span} \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}.$$

Then we say $\mathfrak{S}$ spans V, and $\mathfrak{S}$ is a *spanning* set of V.

0.2 Inner Product

The addition and scalar multiplication define the *algebraic structure* of a vector space. In order to introduce geometry to linear algebra, we can endow geometric structure onto it by defining the concept called *inner product*.

Definition 0.3

An inner product on a vector space V is a function $\langle\ ,\ \rangle$

$$V \times V \to \mathbb{R}$$

satisfying the following properties: let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbb{R}$, we have

$(a).\ \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ — *Symmetry*

$(b).\ \langle \mathbf{u}, r\mathbf{v} + s\mathbf{w} \rangle = r\langle \mathbf{u}, \mathbf{v} \rangle + s\langle \mathbf{u}, \mathbf{w} \rangle$ — *Linearity*

$(c).\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ *and equality is true if and only if* $\mathbf{u} = \mathbf{0}$ — *Positivity*

♣

Once we introduce the inner product, we introduce geometry into vector space. For example

Definition 0.4

(Length of a vector) We can define the length, or the norm, of a vector to be

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

(Distance) Let $\mathbf{u}, \mathbf{v}$ be two vectors. Then their distance is defined to be

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}.$$

♣

Note in the definition of distance, we used both the geometric structure (inner product) and algebraic structrue (subtraction is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1) \cdot \mathbf{v}$).

Example 0.5 In $\mathbb{R}^3$ (and in $\mathbb{R}^n$), the ordinary *dot product*

$$\langle (a^1, a^2, a^3), (b^1, b^2, b^3) \rangle = a^1 b^1 + a^2 b^2 + a^3 b^3$$

is an inner product.

However, there are many other ways one can define inner product on $\mathbb{R}^3$. For example, we can define

$$\langle (a^1, a^2, a^3), (b^1, b^2, b^3) \rangle = 2a^1 b^1 + 3a^2 b^2 + 4a^3 b^3.$$

It is an inner product. In general, let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

be a positive definite matrix. Then

$$\langle (a^1, a^2, a^3), (b^1, b^2, b^3) \rangle = \sum_{i,j=1}^{3} r_{ij} a^i b^j$$

is an inner product.

Example 0.6 We let $R[t]$ be the vector space of all polynomials. Define the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) \, dx$$

More general, if $\rho(x) > 0$ be a positive continuous function, then

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) \rho(x) \, dx$$

is an inner product.

One of the most important inequality in mathematics is called the *Cauchy-Schwarz Inequality*.

Theorem 0.1

Let $\mathbf{u}, \mathbf{v}$ be two vectors. Then we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|,$$

and the equality holds if and only if $\mathbf{u}, \mathbf{v}$ are linearly dependent.

♡

Proof. This is the standard proof. Let t be a real number. Then by the positivity of the inner product, we have

$$\langle \mathbf{u}+t\mathbf{v}, \mathbf{u}+t\mathbf{v}\rangle \geq 0.$$

Using the linearity, we get

$$t^2\langle \mathbf{v},\mathbf{v}\rangle + 2t\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{u}\rangle \geq 0.$$

Since the above inequality is true for any real number t, the discriminant

$$\Delta = 4|\langle \mathbf{u},\mathbf{v}\rangle|^2 - 4\langle \mathbf{u},\mathbf{u}\rangle \cdot \langle \mathbf{v},\mathbf{v}\rangle \leq 0,$$

which proves the inequality. ■

Proof. [Second Proof] Assume that $\mathbf{v} \neq \mathbf{0}$. Then we have

$$\langle \mathbf{u} - \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle}\mathbf{v}, \mathbf{u} - \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\langle \mathbf{v},\mathbf{v}\rangle}\mathbf{v}\rangle \geq 0.$$

Expanding the above inequality, we obtain the Cauchy-Schwarz Inequality. ■

Remark The Cauchy-Schwarz inequality is also called the Cauchy-Bunyakovsky-Schwarz inequality.

The inequality for sums was published by Augustin-Louis Cauchy (1821), while the corresponding inequality for integrals was first proved by Viktor Bunyakovsky (1859). The modern proof of the integral version was given by Hermann Schwarz (1888).

Definition 0.5

Two vectors $\mathbf{u}, \mathbf{v}$ *are called orthogonal, if* $\langle \mathbf{u},\mathbf{v}\rangle = 0$. ♣

Example 0.7 In $\mathbb{R}^3$, the vector $(1,2,3)$ is orthogonal to $(4,-5,2)$ with respect to the dot product, because

$$(1,2,3)\cdot(4,-5,2) = 1\cdot 4 + 2\cdot(-5) + 3\cdot 2 = 0.$$

Example 0.8 Under the inner product

$$\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

The function x and x^2+1 are orthogonal because $p(x)$ is an odd function.

An *orthonormal basis* of an n-dimensional vector space is a set $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ such that

$$\langle \mathbf{v}_i,\mathbf{v}_j\rangle = \begin{cases} 1, & \text{if} \quad i=j \\ 0, & \text{if} \quad i\neq j \end{cases}.$$

We shall introduce the *Kronecker symbol* δ_{ij} as follows

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i=j \\ 0, & \text{if} \quad i\neq j \end{cases}.$$

Under this notation, we have

$$\langle \mathbf{v}_i,\mathbf{v}_j\rangle = \delta_{ij}.$$

0.3 Linear Transformation

Definition 0.6

Given two vector spaces, V and W, a linear transformation T from V to W is a mapping

$$T : V \to W$$

such that

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$. ♣

A linear transformation of a vector space to itself is called an *endomorphism*. One of the most important concept of an endomorphism is its eigenvalue and eigenvector.

Definition 0.7

Let $T : V \to V$ be an endomorphism. Assume that there is a $\mathbf{v} \in V$ and $\mathbf{v} \neq 0$ such that

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

for some complex number λ. Then λ is called an eigenvalue of T and v is an eigenvector of λ. ♣

Note *In the following, we need to prove that the definition of eigenvalue is equivalent to the definition of an eigenvalue of a matrix.*

Let V be a finite dimensional space and let

$$\mathfrak{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$$

be a basis of V. We use the notation $[\mathbf{x}]_{\mathfrak{B}}$ to represent the coordinates of $\mathbf{x} \in V$, that is, when we write x in terms of the linear combination of the basis,

$$\mathbf{x} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n,$$

we have

$$[\mathbf{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Now let $T : V \to W$ be a linear transformation. As above, we assume that $\mathfrak{B}$ is the basis of V, and let $\mathfrak{C} = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ be a basis of W. Then the matrix representation M of T is

$$M = [[T(\mathbf{v}_1)]_{\mathfrak{C}}, \cdots, [T(\mathbf{v}_n)]_{\mathfrak{C}}]$$

in the sense that

$$[T(\mathbf{x})]_{\mathfrak{C}} = M \cdot [\mathbf{x}]_{\mathfrak{B}}.$$

Now we specialize the above result to the following case. Let

$$T : V \to V$$

be a linear transformation from V to itself. Let

$$\mathfrak{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}, \qquad \mathfrak{C} = \{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$$

be two bases. Let M_1, M_2 be the matrix representatives with respect to the two bases respectively. We thus have

$$[T(\mathbf{x})]_{\mathfrak{B}} = M_1 \cdot [\mathbf{x}]_{\mathfrak{B}};$$
$$[T(\mathbf{x})]_{\mathfrak{C}} = M_2 \cdot [\mathbf{x}]_{\mathfrak{C}}.$$

Let A be the invertible matrix such that

$$[\mathbf{x}]_{\mathfrak{B}} = A \cdot [\mathbf{x}]_{\mathfrak{C}}.$$

Such a matrix is called a *transition matrix*. Then

$$[T(\mathbf{x})]_{\mathfrak{B}} = A \cdot [T(\mathbf{x})]_{\mathfrak{C}}.$$

As a result,

$$A \cdot M_2 \cdot [\mathbf{x}]_{\mathfrak{C}} = A \cdot [T(\mathbf{x})]_{\mathfrak{C}} = [T(\mathbf{x})]_{\mathfrak{B}} = M_1 \cdot A \cdot [\mathbf{x}]_{\mathfrak{C}}.$$

Thus we have

$$AM_2 = M_1 A,$$

or

$$M_2 = A^{-1} M_1 A.$$

Thus M_1, M_2 are similar, having the same eigenvalue set.

Now we talk about *orientation and cross product*.

Let $\mathfrak{B}$ and $\mathfrak{C}$ be two bases. We say that $\mathfrak{B}$ and $\mathfrak{C}$ are having the same orientation, if when we write

$$\mathbf{v}_i = \sum_{j=1}^{n} a_{ij} \mathbf{w}_j$$

for $i = 1, \cdots, n$, then we have $\det(A) = \det(a_{ij}) > 0$. They give the opposite orientation if $\det(a_{ij}) < 0$.

Example 0.9 The left hand and the right hand define two opposite orientations of $\mathbb{R}^3$.

Example 0.10 Let

$$\mathfrak{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

and

$$\mathfrak{C} = \{(1,1,0), (1,0,-1), (2,1,3)\}$$

are of the opposite orientation.

Definition 0.8

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be the standard basis of* $\mathbb{R}^3$*. If*

$$\mathbf{u} = \sum_{i=1}^{3} a_i \mathbf{e}_i, \qquad \mathbf{v} = \sum_{j=1}^{3} b_j \mathbf{e}_j$$

are two vectors in $\mathbb{R}^3$*, the cross product of* $\mathbf{u}, \mathbf{v}$ *is given by*

$$\mathbf{u} \times \mathbf{v} = (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3,$$

or we can write

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

♣

The cross product satisfies the following properties:

Lemma 0.1

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* $r \in \mathbb{R}$*. Then*

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- $(r\mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v})$
- $\mathbf{u} \times \mathbf{v} = 0$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent*
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
- $\mathbf{u} \times \mathbf{v}$ *is perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$ *under the usual dot product*
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin\theta$*, where* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*.*
- $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ *gives a right hand orientation to* $\mathbb{R}^3$ *if* $\{\mathbf{u}, \mathbf{v}\}$ *is linearly independent*

♡

As a result, we have the relationship between the inner product and outer product (cross product)

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 + \|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2,$$

which implies the Cauchy-Schwarz Inequality in the three dimensional space.

Definition 0.9

The mixed (or triple) product of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is*

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle.$$

♣

A geometric interpretation of the norm of the cross product is that it is the area of the parallelogram spanned by $\mathbf{u}, \mathbf{v}$. A geometric interpretation of the mixed scalar product is that it is the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

0.4 Lines, planes, and spheres

In this section, we use vector notations to express some basic objects in analytic geometry.

Definition 0.10

The line through $\mathbf{x}_0 \in \mathbb{R}^3$ *and parallel to a vector* $\mathbf{v} \neq 0$ *has the equation*

$$\alpha(t) = \mathbf{x}_0 + t\mathbf{v}.$$

♣

Remark This is a vector notation of parametrization of a line.

Example 0.11 Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ be two points. Then the line through $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ has the equation.

$$\alpha(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1).$$

Definition 0.11

The plane through $\mathbf{x}_0$ *perpendicular to* $\mathbf{n} \neq 0$ *has the equation*

$$\langle \mathbf{x} - \mathbf{x}_0, \mathbf{n} \rangle = 0.$$

♣

Lemma 0.2

Let $\{\mathbf{u}, \mathbf{v}\}$ *be two linearly independent vectors. Then the place through* $\mathbf{x}_0$ *and parallel to the subspace spanned by* $\{\mathbf{u}, \mathbf{v}\}$ *has the equation*

$$[\mathbf{u}, \mathbf{v}, \mathbf{x} - \mathbf{x}_0] = \langle \mathbf{x} - \mathbf{x}_0, \mathbf{u} \times \mathbf{v} \rangle = 0.$$

♡

Definition 0.12

The sphere in $\mathbb{R}^3$ *with center* $\mathbf{m}$ *and radius* $r > 0$ *has equation*

$$\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle = \|\mathbf{x} - \mathbf{m}\|^2 = r^2. \tag{1}$$

♣

Remark Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}.$$

Then we get the usual equation of a sphere

$$(x_1 - m_1)^2 + (x_2 - m_2)^2 + (x_3 - m_3)^2 = r^2.$$

Example 0.12 (Kelvin Transformation)

We consider the equation of a sphere (1). Let $\mathbf{x}_0$ be a point on the sphere, that is, we have

$$\langle \mathbf{x}_0 - \mathbf{m}, \mathbf{x}_0 - \mathbf{m} \rangle = r^2.$$

The Kelvin Transformation is a map

$$K : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x} \mapsto \mathbf{x}_0 + \frac{\mathbf{x} - \mathbf{x}_0}{\|\mathbf{x} - \mathbf{x}_o\|^2}$$

By a straightforward computation, we have $K^2 = id$. Assume

$$\langle K(x) - \mathbf{m}, K(x) - \mathbf{m}\rangle = r^2.$$

We get

$$1 + 2\langle \mathbf{x} - \mathbf{x}_0, \mathbf{x}_0 - \mathbf{m}\rangle = 0.$$

So the Kelvin transformation maps a sphere to a plane.

External Link. *The detailed computation can be found here.*

Example 0.13 (Ptolemy Inequality) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ be four vectors in the Euclidean plane. Then we have

$$\|\mathbf{u} - \mathbf{w}\| \cdot \|\mathbf{v} - \mathbf{x}\| \leq \|\mathbf{u} - \mathbf{v}\| \cdot \|\mathbf{x} - \mathbf{w}\| + \|\mathbf{u} - \mathbf{x}\| \cdot \|\mathbf{v} - \mathbf{w}\|.$$

The equality is valid if and only if these four vectors are concyclic.

External Link. *The Ptolemy Inequality is closely related to the Ptolemy Theorem. For details of the Ptolemy and his theorem, see Wikepedia of Ptolemy Theorem*

0.5 Vector Calculus

In differential geometry, in addition to study functions of several variables. We also need to study vector-valued functions.

We can define derivatives, indefinite integral and definite integrable in similar ways to those of multi-variable functions.

Let V, W be finite dimensional vector spaces. Let

$$F : V \to W$$

be a *differentiable* function, with definition as follows.

Definition 0.13

We fix a basis of V and using that basis, we identify V to $\mathbb{R}^n$. Similarly, and we fix a basis of W and identify it to $\mathbb{R}^m$. Then we can identify $F : V \to W$ by $F : \mathbb{R}^n \to \mathbb{R}^m$. So F is differentiable if and only if F is a differentiable as a mapping of $\mathbb{R}^n \to \mathbb{R}^m$. ♣

External Link. *Here is a video clip for the details of the above definition.*

Let $f : \mathbb{R} \to \mathbb{R}^n$ be a single variable vector-valued function. We can define

$$\frac{df}{dt} = \begin{bmatrix} \frac{df_1}{dt} \\ \vdots \\ \frac{df_n}{dt} \end{bmatrix}$$

if

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Likewise, we can define

$$\int f(t)\,dt, \qquad \int_a^b f(t)\,dt$$

in similar ways.

If $f : \mathbb{R} \to V$ be an abstract vector-valued function, we can identity V with $\mathbb{R}^n$ under a fixed basis, and define the derivative, integral, etc, similarly.

Lemma 0.3

Let $f : \mathbb{R} \to V, g : \mathbb{R} \to V$ and let $\langle\ ,\ \rangle$ be an inner product on V. Then if f and g are differentiable, so is $\langle f, g\rangle$, which is a function of one variable. Moreover, we have

$$\frac{d}{dt}\langle f, g\rangle = \langle \frac{df}{dt}, g\rangle + \langle f, \frac{dg}{dt}\rangle.$$

♡

Similarly, we have

Lemma 0.4

Using the notations as in the above lemma, we have

$$\frac{d}{dt}(f \times g) = \frac{df}{dt} \times g + f \times \frac{dg}{dt}.$$

♡

Both of the above two lemmas can be proved directly. Moreover, we can generalize the above results into the following.

Let V, W, S be vector spaces (probably of infinite dimensional) and let

$$K : V \times W \to S$$

be a map. We say K is *bilinear*, if K is linear with respect to each component.

Both the inner product and cross product are bilinear mappings.

Let $f : \mathbb{R} \to V$, $g : \mathbb{R} \to W$ be differentiable functions. Then the function

$$h(t) = K(f(t), g(t))$$

is differentiable, and

$$\frac{dh}{dt} = K(\frac{df}{dt}, g(t)) + K(f(t), \frac{dg}{dt}).$$

Definition 0.14

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say f is of class C^k, if all derivatives up through order k exist and are continuous.

$f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k if all its (mixed) partial derivatives of up through order k exist

and are continuous. A vector-valued function is of class $\mathcal{C}^k$ if all of its components with respect to a given basis are of class $\mathcal{C}^k$.

If f is of class $\mathcal{C}^k$ for any k, we say f is of $\mathcal{C}^\infty$, or we say f is smooth. ♣

We assume most of the functions we shall study in this course are smooth, or at least of $\mathcal{C}^3$.

Finally, we review the chain rule. Let x be a function of $(u_1, \cdots, u_n)$, and if each u_i are functions of $(v_1, \cdots, v_m)$, say,

$$u_i = u_i(v_1, \cdots, v_m).$$

for $i = 1, \cdots, n$. Then we have the chain rule

$$\frac{\partial x}{\partial v_\alpha} = \sum_{i=1}^{n} \frac{\partial x}{\partial u_i} \cdot \frac{\partial u_i}{\partial v_\alpha} \tag{2}$$

for $\alpha = 1, \cdots, m$.

0.6 Einstein Convention

Definition 0.15. Einstein Convention

When an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index. When an index variable appears only once, it implies that the equation is valid for every value of such an index. ♣

For example, Equation (2) can be written as

$$\frac{\partial x}{\partial v_\alpha} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial u_i}{\partial v_\alpha}.$$

In the above equation, the index i in the right appears twice, so we assume the expression is summing over all possible i. On the other hand, the index α appears only once, so we assume the equation is valid for all range of α.

Example 0.14 Let $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ be matrices. Then the matrix multiplication,

$$C = AB,$$

can be written using the Einstein Convention as

$$c_{ij} = a_{ik} b_{kj}.$$

The Einstein Convention gives another way to express and generalize linear algebra.

Note *Let's discuss the representation of a matrix. In linear algebra, there are three ways to represent a matrix*

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

where $\mathbf{a}_1, \cdots, \mathbf{a}_n$ *are column vectors. The easiest way to represent a matrix is to a capital letter, say* A*. But this would contain the least amount of information about the matrix. On the other extreme, if we represent a matrix by providing all the details, it would be too clumsy to write down.*

Here we give the fourth method of representing a matrix, by writing it as (a_{ij})*, which takes care of both simplicity and information.*

Example 0.15 Prove the associativity of matrix multiplication.

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$ be matrices. Let $D = (d_{ij})$ be the matrix $D = AB$. In terms of the Einstein Convention, $D = AB$ is equivalent to

$$d_{ij} = a_{ik}b_{kj}. \tag{3}$$

Here the index k is called a *dummy* index in the sense that we can replace it with other indices without changing the equations:

$$d_{ij} = a_{ik}b_{kj} = a_{it}b_{tj} = a_{i\alpha}b_{\alpha j}. \tag{4}$$

Now let $C = (c_{ij})$, $E = BC = (e_{ij})$, $F = (AB)C = (f_{ij})$ and $G = A(BC) = (g_{ij})$. Then the entries for $(AB)C = DC$ would be

$$f_{ij} = d_{ik}c_{kj} = d_{it}c_{tj}.$$

From (2.1), we know that $d_{it} = a_{ik}b_{kt}$. Thus

$$f_{ij} = a_{ik}b_{kt}c_{tj}.$$

The reason we use t as the dummy index in (2.1) is because k has been used in (3) so we need to use a different one, keeping indices repeated at most twice.

Similarly, we have

$$g_{ij} = a_{it}b_{tk}c_{kj}.$$

Thus $f_{ij} = g_{ij}$ and hence

$$(AB)C = A(BC),$$

proving the associativity. ■

Example 0.16 Using the Einstein Convention to prove the following version of the Cauchy inequality. Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$. Then

$$|\mathbf{x} \cdot \mathbf{y}|^2 \le \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2.$$

Proof. Using the Einstein Convention, we can write

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_i y_i.$$

Thus

$$|\mathbf{x} \cdot \mathbf{y}|^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \left(\sum_{i=1}^{n} x_i y_i\right) \cdot \left(\sum_{j=1}^{n} x_j y_j\right) = x_i y_i x_j y_j.$$

On the other hands, we can write

$$\|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2 = \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{j=1}^{n} y_j^2\right) = x_i^2 y_j^2.$$

Thus the Cauchy inequality, written under the Einstein Convention, is

$$x_i^2 y_j^2 - x_i y_i x_j y_j = \frac{1}{2}(x_i^2 y_j^2 + x_j^2 y_i^2) - x_i y_i x_j y_j = \frac{1}{2}(x_i y_j - x_j y_i)^2 \geq 0.$$

This completes the proof. ∎

Note *If $n = 3$, then we can define*

$$(\mathbf{x}, \mathbf{y}) \mapsto (x_2 y_3 - x_2 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) = \mathbf{x} \times \mathbf{y}$$

which is the cross product. If $n \neq 3$, then the vector $(x_i y_j - x_j y_i)$ for $i < j$ is of dimension $n(n-1)/2 \neq 3$. This explains why we can only define the cross product in 3 dimensional vector space. A more general algebraic product, called wedge product, will be used in any dimensional vector spaces to catch in the excess of the Cauchy inequality.

External Link. *As fun reading, you can find the Shoelace Formula in the Wikipedia, which is related to both the cross product and wedge product.*

Chapter 1 Calculus on Euclidean Space

Introduction

- *Calculus on Euclidean space*
- *Tangent Vectors and tangent space*
- *The space of differential forms*
- *The wedge product*
- *The differential operator d*
- *The differential of a mapping*

1.1 Euclidean Space

Definition 1.1

Euclidean 3-space $\mathbb{R}^3$ *is the set of all ordered triples of real numbers. Such a triple* $\mathbf{p} = (p_1, p_2, p_3)$ *is called a* point *of* $\mathbb{R}^3$.

By last chapter, $\mathbb{R}^3$ is a vector space.

Definition 1.2

On $\mathbb{R}^3$*, there are three natural real-valued functions* x, y, z*, defined by*

$$x(\mathbf{p}) = p_1, \quad y(\mathbf{p}) = p_2, \quad z(\mathbf{p}) = p_3.$$

These functions are called natural coordinate functions of $\mathbb{R}^3$.

Remark We shall also use index notation for these functions, writing

$$x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

Definition 1.3

A real-valued function f *on* $\mathbb{R}^3$ *is differentiable (or infinitely differentiable, or smooth, or of class* $\mathcal{C}^\infty$*) provided all partial derivatives of* f*, of all orders, exist and are continuous.*

As we know from the previous chapter, the space of smooth functions forms a vector space, that is, let f, g be two smooth functions of $\mathbb{R}^3$ and let $\lambda \in \mathbb{R}$, we have

$$(f+g)(\mathbf{p}) = f(\mathbf{p}) + g(\mathbf{p}), \quad (\lambda f)(\mathbf{p}) = \lambda f(\mathbf{p}).$$

In addition, we have

$$(fg)(\mathbf{p}) = f(\mathbf{p})g(\mathbf{p}).$$

The space of smooth functions, with respect to the three operations: addition, scalar multiplication, and the multiplication forms an *algebra*.

1.2 Tangent Vectors

Definition 1.4

A tangent vector, or a vector $\mathbf{v_p}$ *to* $\mathbb{R}^3$ *consists of two points of* $\mathbb{R}^3$*: its vector part* $\mathbf{v}$ *and its point of application* $\mathbf{p}$.

♣

Definition 1.5

Let $\mathbf{p}$ *be a point of* $\mathbb{R}^3$*. The set* $T_\mathbf{p}(\mathbb{R}^3)$ *consisting of all tangent vectors that have* $\mathbf{p}$ *as point of application is called the tangent space of* $\mathbb{R}^3$ *at* $\mathbf{p}$.

♣

Note *Tangent space is a vector space.*

Definition 1.6

A tangent vector field, or a vector field V *on* $\mathbb{R}^3$ *is a function that assigns to each point* $\mathbf{p}$ *of* $\mathbb{R}^3$ *a tangent vector* $V(\mathbf{p})$ *to* $\mathbb{R}^3$ *at* $\mathbf{p}$.

♣

Note *Vector field is one of the most important concepts in differential geometry. By the above definition, a vector field is just a vector valued function. This is because* $\mathbb{R}^3$ *is a flat space, and hence there are global basis under which all tangent spaces can be identified as* $\mathbb{R}^3$*. In general, a vector field defines a different type of "functions" comparing to the traditional one.*

Remark By definition, a vector field doesn't have to be smooth. However, in this course, we always assume it is smooth (or at least of $\mathcal{C}^3$) when regarding it as a vector-valued function.

The domain of a vector field doesn't have to be on the whole $\mathbb{R}^3$: it could be an open set of $\mathbb{R}^3$, or a curve or a surface in $\mathbb{R}^3$. In the latter to cases, we say that the vector field is *along the curve or surface*.

The space of vector fields is obviously a vector space. However, it has finer structure than that. It is a *module* over the algebra of smooth functions.

There are two operations on the space of vector fields: addition and scalar multiplication. Let V, W be two vector fields such that $V = v_i U_i$, $W = w_i U_i$. Let $\lambda \in \mathbb{R}$. Then we can define

$$V + W = \sum_i (v_i + w_i) U_i$$

$$\lambda V = \sum (\lambda v_i) U_i.$$

Moreover, let f be a smooth function of $\mathbb{R}^3$. Then we can define

$$(fV)(\mathbf{p}) = f(\mathbf{p}) V(\mathbf{p})$$

for all $\mathbf{p}$. Of course, such kind of multiplication can be localized to the case when V and f are only defined on a subset of $\mathbb{R}^3$.

Note *The scalar multiplication coincides with the above multiplication by regarding a scalar as a constant function.*

Note *We can also say that the space of vector fields is a module over the ring of smooth functions. But the algebra of smooth functions carries more structure than the ring structure of smooth functions: it has the additional scalar multiple structure. Therefore it is better to say the module over the algebra of smooth functions than that over the ring of smooth functions.*

Definition 1.7

Let U_1, U_2 *and* U_3 *be the vector fields on* $\mathbb{R}^3$ *such that*

$$U_1(\mathbf{p}) = (1,0,0)_{\mathbf{p}}$$

$$U_2(\mathbf{p}) = (0,1,0)_{\mathbf{p}}$$

$$U_3(\mathbf{p}) = (0,0,1)_{\mathbf{p}}$$

for each $\mathbf{p}$ *of* $\mathbb{R}^3$. *We call* $\{U_1, U_2, U_3\}$ *the natural frame field of* $\mathbb{R}^3$.

Remark For fixed point, $\{U_1, U_2, U_3\}$ provides the standard basis of $\mathbb{R}^3$, usually expressed as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The following result is a generalization of what we have learned in linear algebra.

Lemma 1.1

If V *is a vector field of* $\mathbb{R}^3$, *then there are three uniquely determined real-valued functions* v_1, v_2, v_3 *on* $\mathbb{R}^3$ *such that*

$$V = v_1U_1 + v_2U_2 + v_3U_3.$$

These three functions are called Euclidean coordinate functions of V.

Proof. For fixed $\mathbf{p} \in \mathbb{R}^3$, $V(\mathbf{p})$ defines a *vector* in $\mathbb{R}^3$, therefore there is unique numbers $v_1(\mathbf{p})$, $v_2(\mathbf{p})$, and $v_3(\mathbf{p})$ such that

$$V(\mathbf{p}) = v_1(\mathbf{p})U_1(\mathbf{p}) + v_2(\mathbf{p})U_2(\mathbf{p}) + v_3(\mathbf{p})U_3(\mathbf{p}).$$

Thus

$$V = v_iU_i$$

by *definition*. ∎

1.3 Directional Derivatives

Definition 1.8

Let f *be a differentiable real-valued function on* $\mathbb{R}^3$, *and let* $v_{\mathbf{p}} \in T_{\mathbf{p}}(\mathbb{R}^3)$ *be a tangent vector to* $\mathbb{R}^3$. *Then the number*

$$v_{\mathbf{p}}[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))|_{t=0}$$

is called the *derivative* *of f with respect to $v_{\mathbf{p}}$*

Note *It is called the directional derivative because $\mathbf{p}+t\mathbf{v}$ for non-negative real number t represents a ray starting from $\mathbf{p}$ in the direction $\mathbf{v}$. We have encountered directional derivative in Calculus. Here the emphasis is that "vector" (which is an algebraic concept) can be identified as a "derivative" (which is a calculus concept).*

Lemma 1.2

If $v_{\mathbf{p}} = (v_1, v_2, v_3)$ is a tangent vector to $\mathbb{R}^3$, then

$$v_{\mathbf{p}}[f] = \sum v_i \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

Proof. Let $\mathbf{p} = (p_1, p_2, p_3)$; then

$$\mathbf{p} + t\mathbf{v} = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3).$$

We then

$$v_{\mathbf{p}}[f] = \frac{d}{dt}(f(\mathbf{p}+t\mathbf{v}))|_{t=0} = \sum_i \frac{\partial f}{\partial x_i}(\mathbf{p})v_i.$$

■

Example 1.1 Let $f(x,y,z) = x^2yz$. Let $\mathbf{p} = (1,1,0)$ and $\mathbf{v} = (1,0,-3)$. Then

$$\frac{\partial f}{\partial x} = 2xyz, \qquad \frac{\partial f}{\partial y} = x^2 z, \qquad \frac{\partial f}{\partial z} = x^2 y.$$

Thus

$$\frac{\partial f}{\partial x}(\mathbf{p}) = 0, \qquad \frac{\partial f}{\partial y}(\mathbf{p}) = 0, \qquad \frac{\partial f}{\partial z}(\mathbf{p}) = 1.$$

Thenfore

$$v_{\mathbf{p}}[f] = 0 + 0 + 1 \cdot (-3) = -3.$$

Theorem 1.1

Let f and g be functions on $\mathbb{R}^3$, $v_{\mathbf{p}}$ and $w_{\mathbf{p}}$ tangent vectors, a and b numbers. Then

- $(av_{\mathbf{p}} + bw_{\mathbf{p}})[f] = av_{\mathbf{p}}[f] + bw_{\mathbf{p}}[f]$;
- $v_{\mathbf{p}}(af + bg) = av_{\mathbf{p}}[f] + bv_{\mathbf{p}}[g]$;
- $v_{\mathbf{p}}[fg] = v_{\mathbf{p}}[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot v_{\mathbf{p}}[g]$.

Proof. Only the 3rd equation is new, which can be proved using Lemma 1.2: We have

$$v_{\mathbf{p}}[fg] = v_i \frac{\partial (fg)}{\partial x_i}(\mathbf{p}) = v_i f(\mathbf{p}) \frac{\partial g}{\partial x_i}(\mathbf{p}) + v_i g(\mathbf{p}) \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

By definition, we have

$$v_{\mathbf{p}}[fg] = v_{\mathbf{p}}[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot v_{\mathbf{p}}[g].$$

1.4 Curves in $\mathbb{R}^3$

One of the fundamental questions in curve theory is: how to define a curve? In Euclidean geometry, only two kinds of curves are studied: straight line and circle. In analytic geometry, we study parabola, ellipse, and hyperbola. These curves have quite explicit geometric meanings. For example, an ellipse is the the set of all points in a plane such that the sum of the distances from two fixed points (foci) is constant. If we want to study more general curves, we should not expect them have clear geometric meanings.

In differential geometry, we define a curve in $\mathbb{R}^3$ by a function

$$\alpha : I \to \mathbb{R}^3, \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)),$$

where I is an open interval. In order to use calculus, we usually assume that such a function is smooth.

Definition 1.9

A curve in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$ from an open interval into $\mathbb{R}^3$. ♣

We shall give a couple of examples of curves.

Example 1.2 (Straight Line) A *straight line* can be expressed best using the vector notations. Let $\mathbf{p}$, $\mathbf{q}$ be two vectors and let $\mathbf{q} \neq 0$. Then we can use

$$\alpha(t) = \mathbf{p} + t\mathbf{q}$$

to represent a curve with direction $\mathbf{q}$.

Example 1.3 (Helix) The parameter equations for a circle (in $\mathbb{R}^3$) can be expressed by

$$t \mapsto (a\cos t, a\sin t, 0).$$

If we allow this curve to rise , then we obtain a *helix* $\alpha : \mathbb{R} \to \mathbb{R}^3$, given by the formula

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

where $a > 0$ and $b \neq 0$.

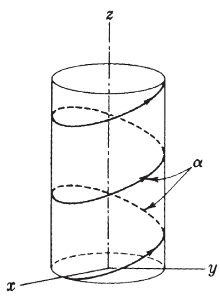


FIG. 1.7

Definition 1.10

Let $\alpha : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each number $t \in I$, the velocity vector of α at t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right)_{\alpha(t)}$$

at the point $\alpha(t)$ in $\mathbb{R}^3$.

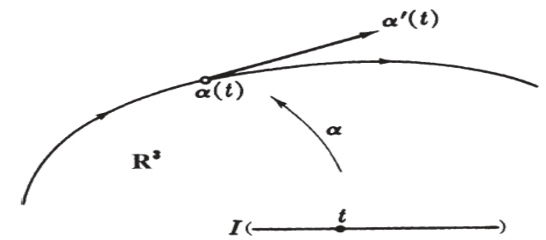


FIG. 1.9

;

Example 1.4 For the helix,

$$\alpha(t) = (a\cos t, a\sin t, bt),$$

the velocity vector is

$$\alpha'(t) = (-a\sin t, a\cos t, b)_{\alpha(t)}.$$

Definition 1.11

Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $h : J \to I$ is a differentiable function on an open interval J, then the composition function

$$\beta = \alpha(h) : J \to \mathbb{R}^3$$

is a curve called a reparametrization of α by h.

Note *The above definition is a key concept. See the next lemma.*

Lemma 1.3

If β is the reparametrization of α by h, then

$$\beta'(s) = \frac{df}{ds} \cdot \alpha'(h(s)).$$

Proof. This is a straightforward application of the chain rule.

Lemma 1.4

Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

We end up this section by a discussion of *dual space*.

Dual Space

Definition 1.12

Given any vector space V, the dual space V^* is defined as the set of all linear transformations $\varphi : V \to \mathbb{R}$. The dual space is a vector space by the following definition of addition and scalar multiplication.

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a(\varphi(x))$$

for all $\varphi, \psi \in V^*$ and $a \in \mathbb{R}$, $x \in V$. Elements of V^* is called a covector, or linear functional, or a one-form. ♣

Example 1.5 On $\mathbb{R}^n$, any linear function

$$\ell(\mathbf{x}) = c_1 x_1 + \cdots + c_n x_n,$$

where $\mathbf{x} = (x_1, \cdots, x_n)$ and $c_1, \cdots, c_n$ being real numbers, is a liner functional.

Example 1.6 Let $\mathcal{C}([0,1])$ be the vector space of continuous functions over $[0,1]$. Then

$$f \mapsto \int_0^1 f(x)dx$$

defines a linear functional.

Example 1.7 Let $\mathbf{p} \in \mathbb{R}^3$. Let $\mathbf{v_p}$ be a vector on $T_{\mathbf{p}}(\mathbb{R}^3)$. Then the directional derivative

$$f \mapsto \mathbf{v_p}[f]$$

is a linear functional on the vector space of differentiable functions.

External Link. *Here is a good video of the dual space. The first 8 minutes is useful, and the last part is beyond the scope of this course.*

Theorem 1.2. (The Riesz Representation Theorem)

Let V be a finite dimensional vector space and let $\langle\ ,\ \rangle$ be an inner product of V. Let $\mathbf{x} \in V$. Then it defines a linear functional $\ell_{\mathbf{x}}$ such that $\ell_{\mathbf{x}}(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y}\rangle$ for any $\mathbf{y} \in V$; conversely, let ℓ be a linear functional, then there is a unique $\mathbf{x} \in V$ such that $\ell(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y}\rangle$ for any $\mathbf{y} \in V$. ♡

Remark In order words, every linear functional can be *represented*, through a fixed inner product, as an element of the vector space.

Remark We elaborate the Riesz Representation Theorem in the context of the vector space $\mathbb{R}^n$ with the dot product. Let $\mathbf{x} \in \mathbb{R}^n$, and let ℓ be a linear functional. By the above theorem, there is a vector $\mathbf{c}$ such that

$$\ell(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = c_1 x_1 + \cdots + c_n x_n.$$

In this way, the dual space of $\mathbb{R}^n$ can be identified to $\mathbb{R}^n$.

Note *There is an infinite dimensional version of the Riesz Representation Theorem on normed vector space, but the linear functional in question needs to be replaced by bounded linear functional.*

External Link. *The linear algebra over infinite dimensional vector spaces is called Functional Analysis.*

1.5 1-forms

Definition 1.13

A 1-form ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is,

$$\phi(a\mathbf{v} + b\mathbf{w}) = a\phi(\mathbf{v}) + b\phi(\mathbf{w})$$

for any number a, b and tangent vectors $\mathbf{v}, \mathbf{w}$ at the same point of $\mathbb{R}^3$.[a]

[a] Recall in Definition 1.4, a vector on $\mathbb{R}^3$ is a pair $(\mathbf{p}, \mathbf{v})$, where $\mathbf{p} \in \mathbb{R}^3$ and $\mathbf{v}$ is the vector part.

♣

As before, the space of 1-forms is a module over the algebra of smooth functions. Let φ, ψ be two 1-forms; let $\mathbf{v}$ be a vector on $\mathbb{R}^3$; let $\lambda \in \mathbb{R}$. Then we can define

$$(\varphi + \psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v}); \qquad (\lambda\varphi)(\mathbf{v}) = \lambda\varphi(\mathbf{v}).$$

Note $\varphi(\mathbf{v})$ is a smooth function on $\mathbb{R}^3$. Let f be a smooth function. Let $\mathbf{v_p}$ be the vector part of $\mathbf{v}$.

$$(f\varphi)(\mathbf{v_p}) = f(\mathbf{p})\varphi(\mathbf{v_p}).$$

In fact, there is a natural way to extend a 1-from as a function over vector fields. A 1-form is a linear functional in two ways: first, it is a linear functional over the vector space of vector fields, that is, if φ is a 1-form, for any vector field $\mathbf{v}$, $(\varphi(\mathbf{v}))(\mathbf{p}) = \varphi(\mathbf{v_p})$ is a smooth function; second, for any fixed point $\mathbf{p}$, φ is a linear functional over $T_{\mathbf{p}}(\mathbb{R}^3)$.

Definition 1.14

If f is a differentiable function on $\mathbb{R}^3$. Then df is a 1-form defined by

$$df(\mathbf{v_p}) = \mathbf{v_p}[f].$$

♣

Example 1.8 1-forms on $\mathbb{R}^3$: by the above definition, we can define 1-froms dx_1, dx_2, dx_3. Let

$$\mathbf{v_p} = v_i U_i.$$

Then by definition,

$$dx_i[\mathbf{v_p}] = \mathbf{v_p}[x_i] = v_i.$$

Let's consider the 1-form

$$\psi = f_i dx_i,$$

where f_i are functions. Then

$$\psi[\mathbf{v_p}] = f_i dx_i[\mathbf{v_p}] = f_i(\mathbf{p})v_i.$$

Definition 1.15. Dual Basis

Let V be a vector space and let $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ be a basis of V. $(\mathbf{f}_1, \cdots, \mathbf{f}_n)$ is called the dual basis of $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$, if $\mathbf{f}_i \in V^*$, and if

$$\mathbf{f}_i(\mathbf{e}_j) = \delta_{ij}.$$

♣

Theorem 1.3. Dual Basis Theorem

Using the above notations, then (dx_1, dx_2, dx_3) is the dual basis of (U_1, U_2, U_3).

♡

Proof. We have

$$dx_i(U_j) = U_j[x_i] = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

■

Corollary 1.1

If f is a differentiable function on $\mathbb{R}^3$, then

$$df = \frac{\partial f}{\partial x_i} dx_i = f_i dx_i.$$

♡

Note *As we have seen, we use f_i to represent $\frac{\partial f}{\partial x_i}$. In differential geometry, this would greatly simplify complicated computations. In general, whether f_i is $\frac{\partial f}{\partial x_i}$ or an arbitrary function depends on the context.*

Proof. By definition, $df[\mathbf{v_p}] = \mathbf{v}_p[f]$. But $\mathbf{v}_p[f] = v_i f_i(\mathbf{p}) = (f_i dx_i)[\mathbf{v_p}]$.

■

From the definition of df, we observed that we can regard d as an operator, that would send a function f to a 1-form df. Such an operator is called a *differential operator*, which plays one of the center role in differential geometry.

Lemma 1.5

Let fg be the product of differentiable functions f and g on $\mathbb{R}^3$. Then

$$d(fg) = gdf + fdg.$$

♡

Proof. We have

$$d(fg) = (fg)_i dx_i = (gf_i + fg_i)dx_i = gf_i dx_i + fg_i dx_i = g\,df + f dg.$$

■

Lemma 1.6

Let f be a function on $\mathbb{R}^3$ and let $h : \mathbb{R} \to \mathbb{R}$ be a function of single variable. Then

$$d(h(f)) = h'(f)df.$$

♡

Proof.

$$d(h(f)) = (h(f))_i dx_i = h'(f)f_i dx_i = h'(f)df.$$

■

Example 1.9 Let f be the function

$$f(x, y) = (x^2 - 1)y + (y^2 + 2)z.$$

Then

$$df = 2xydx + (x^2 + 2yz - 1)dy + (y^2 + 2)dz.$$

As a result, we have

$$df[\mathbf{v}] = 2xyv_1 + (x^2 + 2yz - 1)v_2 + (y^2 + 2)v_3.$$

Thus

$$df[\mathbf{v_p}] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_3^2 + 2)v_3.$$

We also have

$$\mathbf{v_p}[f] = 2p_1p_2v_1 + (p_1^2 + 2p_2p_3 - 1)v_2 + (p_3^2 + 2)v_3.$$

This verifies $df[\mathbf{v_p}] = \mathbf{v_p}[f]$.

1.6 Differential Forms

The space of differential 1-forms is a vector space, or more precisely, it is a module over the algebra of smooth functions. To get more information from tangent spaces where the space of differential 1-forms are dual spaces of them at each point, we shall define *multiplication* of differential 1-forms. Since all differential 1-forms are generated by dx_1, dx_2, dx_3, we just need to define their multiplications.

What is $dx_i dx_j$, or we called the *wedge* product $dx_i \wedge dx_j$ of them? We don't know at this moment. But we shall assume that

$$dx_i dx_j = -dx_j dx_i$$

for $1 \leq i, j, \leq 3$. Obviously, this would create a new kind of algebra. For multiplication of real numbers, we have commutativity, which means, for any two real numbers a, b, we have $ab = ba$. On the other hand, for two $n \times n$ matrics A, B, in general, we have $AB \neq BA$. The property for the multiplication of 1-forms are different from both of the above two. It is called *skew commutativity*. The algebra defined by the skew commutativity leads to the so-called *exterior algebra*.

A first observation on the definition of the wedge product reveals that, since $dx_i \wedge dx_i = -dx_i \wedge dx_i$, we must have $dx_i \wedge dx_i = 0$. A quick counting shows that the only non-zero independent products would be $dx_1 dx_2, dx_1 dx_3$ and $dx_2 dx_3$.

In general, we can define the whole system of p-forms. we have already encountered 0-forms, which are smooth functions, and 1-forms. Taking multiplication of dx_i with dx_j, we can define the space of two forms to be generated by $dx_1 dx_2, dx_1 dx_3$ and $dx_2 dx_3$ over smooth functions, that is, all two forms can be expressed by

$$f dx_1 dx_2 + g dx_1 dx_3 + h dx_2 dx_3,$$

where f, g, h are functions.

We can define the 3-forms in an obvious way: all three forms have the expressions

$$f dx_1 dx_2 dx_3,$$

where f is a function.

In the high dimensional case, we can define the p-forms for $p > 3$. However, on $\mathbb{R}^3$, all higher differential forms would be zero: consider, for example, a 4-form $dx_i dx_j dx_k dx_l$. Since the space is of 3 dimensional, at least two of the indices must be the same. By skew commutativity, all 4-forms must be zero.

Example 1.10 Compute the Wedge products

(1). Let

$$\phi = xdx - ydy, \qquad \psi = zdx + xdz.$$

Then

$$\begin{aligned} \phi \wedge \psi &= (xdx - ydy) \wedge (zdx + xdz) \\ &= xzdxdx + x^2 dxdz - yzdydx - xydydz \\ &= yzdxdy + x^2 dxdz - xydydz. \end{aligned}$$

(2). Let $\theta = zdy$. Then

$$\theta \wedge \phi \wedge \psi = -x^2 z \, dxdydz.$$

(3). Let $\eta = ydxdz + xdydz$. Then

$$\begin{aligned} \phi \wedge \eta &= (xdx - ydy) \wedge (ydxdz + xdydz) \\ &= (x^2 + y^2) \, dxdydz. \end{aligned}$$

Note *It should be clear from these examples that the wedge product of a p-form and a q-form is* *a $(p+q)$-form. Thus such a product is automatically zero whenever $p+q>3$.*

Lemma 1.7

Let ϕ, ψ be 1-forms. Then

$$\phi \wedge \psi = -\psi \wedge \phi.$$

Proof. Let $\phi = f_i dx_i, \psi = g_i dx_i$. Then

$$\phi \wedge \psi = f_i dx_i g_j dx_j = f_i g_j dx_i dx_j = -f_i g_j dx_j dx_i = -\psi \wedge \phi.$$

Remark The space of any p-forms forms a module over smooth functions[1]. However, given that a p-form wedge a q-form to be a $(p+q)$-form, we can take the *direct sum* of the modules of all p-forms. Obviously, this would give us a module over functions where the wedge product is well defined.

In what follows we will define arguably the most important concept in differential geometry.

Definition 1.16

If $\phi = f_i dx_i$ is a 1-form on $\mathbb{R}^3$. The exterior derivative, or differential, of ϕ is the 2-form

$$d\phi = df_i \wedge dx_i.$$

Example 1.11 If

$$\phi = f_i dx_i = f_1 dx_1 + f_2 dx_2 + f_3 dx_3,$$

then we have

$$d\phi = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 dx_3.$$

Thus if we identify ϕ to a vector-valued function $E = (f, f_2, f_3)$, then $d\phi$ can be identified as $\mathbf{curl}(E)$. In this sense, $d\phi$ generalize the **curl** operator.

Theorem 1.4

Let f, g be functions and ϕ, ψ be 1-forms. Then

1. $d(fg) = df\, g + f\, dg$;
2. $d(f\phi) = df \wedge \phi + f\, d\phi$;
3. $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$[a].

[a] This is more a definition than a property of the differential operator.

Proof. Property (1) is just the product rule We proved in Lemma 1.5. To prove (2), we let $\phi = f_i dx_i$. Then

$$d(f\phi) = d(ff_i) \wedge dx_i = df \wedge f_i dx_i + f df_i \wedge dx_i = df \wedge \phi + f\, d\phi.$$

[1] For any p, even if $p > 3$ or $p < 0$, where we defined the module to be 0.

Property (3) is, straightly speaking, a definition rather than a property, since we have never defined the differential of 2-forms before. Nevertheless, let's work on it. First,

$$d(\phi \wedge \psi) = d(f_i g_j dx_i dx_j).$$

As in the case of 1-forms, we define

$$d(f_i g_j dx_i dx_j) = d(f_i g_j) \wedge dx_i \wedge x_j.$$

We then have

$$\begin{aligned} d(\phi \wedge \psi) &= df_i g_j \wedge dx_i \wedge dx_j + f_i dg_j \wedge dx_i \wedge dx_j \\ &= df_i \wedge dx_i \wedge g_j dx_j - f_i \wedge dx_i \wedge dg_j \wedge dx_j \\ &= d\phi \wedge \psi - \phi \wedge d\psi. \end{aligned}$$

■

Example 1.12 Let

$$\phi = f_1 dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2$$

be a 2-form. Then

$$d\phi = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 dx_2 dx_3.$$

Thus if we identify $E = (f_1, f_2, f_3)$. Then $d\phi$ can be identify to $\mathbf{div}(E)$.

Example 1.13 Let f be a function, then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

can be identified as ∇f.

Example 1.14 If we identify $\phi = f_i dx_i$ to $\mathbf{u}$ and $\psi = g_i dx_i$ to $\mathbf{v}$, then $\phi \wedge \psi$ can be identified to $\mathbf{u} \times \mathbf{v}$.

✍ **Exercise 1.1** Can you use exterior algebra to define the dot product?

1.7 Mappings

In this section we discuss functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $n = 3$ and $m = 1$, this is just a function on $\mathbb{R}^3$. In the other extreme, if $n = 1$ and $m = 3$, then this is a single variable $\mathbb{R}^3$-valued function, and by the previous sections, they can be used to represent curves in $\mathbb{R}^3$. All of these functions have been studied in Calculus, but in this section, we shall study them using the idea of *linearization*.

Recall that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 1.17

Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$, let $f_1, \cdots, f_m$ denote the real-valued function on $\mathbb{R}^n$

such that

$$F(\mathbf{p}) = (f_1(\mathbf{p}), f_2(\mathbf{p}), \cdots, f_m(\mathbf{p}))$$

for all points $\mathbf{p} \in \mathbb{R}^n$. *These functions are called the Euclidean coordinate functions of* F, *and we can write* $F = (f_1, \cdots, f_m)$.

The functions F *is differentiable provided its coordinate functions are differentiable in the usual sense. A differentiable function* $F : \mathbb{R}^n \to \mathbb{R}^m$ *is called a mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$. ♣

Definition 1.18

If $\alpha : I \to \mathbb{R}^n$ *is a curve in* $\mathbb{R}^n$ *and* $F : \mathbb{R}^n \to \mathbb{R}^m$ *is a mapping, then the composition function* $\beta = F(\alpha) : I \to \mathbb{R}^m$ *is a curve in* $\mathbb{R}^m$ *called the image of* α *under* F. ♣

Note *Let* $\mathscr{B}$ *be the set of all curves in* $\mathbb{R}^n$ *and let* $\mathscr{C}$ *be the set of all curves in* $\mathbb{R}^m$. *Then a mapping* $F : \mathbb{R}^n \to \mathbb{R}^m$ *induces a map* $\mathscr{B} \to \mathscr{C}$.

Definition 1.19

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ *be a mapping. If* $\mathbf{v}$[a] *is tangent vector to* $\mathbb{R}^n$ *at* $\mathbf{p}$. *Let* $F_*(\mathbf{v})$ *be the initial velocity of the curve* $t \mapsto F(\mathbf{p} + t\mathbf{v})$. *The resulting function* F_* *sends a tangent vectors to* $\mathbb{R}^n$ *to tangent vectors to* $\mathbb{R}^m$, *and is called the tangent map of* F.

[a] If $\mathbf{v} = 0$, the straight line $\mathbf{p} + t\mathbf{v}$ is degenerated to a point $\mathbf{p}$. But the definition is still valid. ♣

Proposition 1.1

Let $F = (f_1, \cdots, f_m)$ *be a mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$. *If* $\mathbf{v}$ *is a tangent vector to* R^n *at* $\mathbf{p}$, *then*

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], \cdots, \mathbf{v}[f_m])$$

at $F(\mathbf{p})$. ♠

Proof. We take $m = 3$ for simplicity. By definition, the curve $t \mapsto F(\mathbf{p} + t\mathbf{v})$ can be written as

$$\beta(t) = F(\mathbf{p} + t\mathbf{v}) = (f_1(\mathbf{p} + t\mathbf{v}), f_2(\mathbf{p} + t\mathbf{v}), f_3(\mathbf{p} + t\mathbf{v})).$$

By definition, we have $F_*(\mathbf{v}) = \beta'(0)$. To get $\beta'(0)$, we take the derivatives, at $t = 0$, of the coordinate functions of β. But

$$\frac{d}{dt}(f_i(\mathbf{p} + t\mathbf{v}))|_{t=0} = \mathbf{v}[f_i].$$

Thus

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], \mathbf{v}[f_2], \mathbf{v}[f_3])|_{\beta(0)},$$

where $\beta(0) = F(\mathbf{p})$.

Let $\mathbf{p} \in \mathbb{R}^n$. Then we have the linear transformation

$$F_{*\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to T_{F(\mathbf{p})}(\mathbb{R}^m)$$

called the tangent map of F at $\mathbf{p}$.

Corollary 1.2

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then at each point $\mathbf{p}$ of $\mathbb{R}^n$, the tangent map $F_{*\mathbf{p}} : T_{\mathbf{p}}(\mathbb{R}^n) \to T_{F(\mathbf{p})}(\mathbb{R}^m)$ is a linear transformation. ♡

Note *For any nonlinear function F, we can define a semi-linear function $F_{*\mathbf{p}}$, where for fixed $\mathbf{p}$, the function is a linear transformation. But the function $F_* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, $(\mathbf{p}, \mathbf{v}) \mapsto F_{*\mathbf{p}}(\mathbf{v})$ is nonlinear with respect to $\mathbf{p}$.*

Note *Let $f(t)$ be a function of single variable. Then $f_* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(t, s) \mapsto sf'(t)$ is the tangent map of f. Such a tangent map can be identified to the derivative $f'(t)$.*

Corollary 1.3

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$. ♡

Corollary 1.4

If $F = (f_1, \cdots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$F_*(U_j(\mathbf{p})) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(\mathbf{p})\bar{U}_i(F(\mathbf{p})),$$

where $\{\bar{U}_i\}$, for $i = 1, \cdots, m$ are natural frame fields of $\mathbb{R}^m$. ♡

Definition 1.20

The matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{p})\right)_{1\le i\le m, 1\le j\le n}$$

is called the Jacobian matrix of F at $\mathbf{p}$. ♣

In terms of matrix notations, we have

$$F_*[U_1(\mathbf{p}), \cdots, U_n(\mathbf{p})] = [\bar{U}_1(F(\mathbf{p})), \cdots, \bar{U}_m(F(\mathbf{p}))] \cdot J.$$

Definition 1.21

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is regular provided that at every point $\mathbf{p}$ of $\mathbb{R}^n$ the tangent map $F_{*\mathbf{p}}$ is one-to-one. ♣

Remark By linear algebra, the following are equivalent

(1) $F_{*\mathbf{p}}$ is one-to-one.

(2) $F_{*\mathbf{p}}(\mathbf{v}_\mathbf{p}) = 0$ implies $\mathbf{v}_\mathbf{p} = 0$.

(3) The Jacobian matrix of F at $\mathbf{p}$ has rank n, the dimension of the domain $\mathbb{R}^n$ of F.

Remark If $m = n$, then we know that, by the Invertible Matrix Theorem, that $F_{*\mathbf{p}}$ is one-to-one if and only if it is onto.

Definition 1.22

Let $\mathscr{U}, \mathscr{V}$ be two open sets of $\mathbb{R}^n$. We say that $\mathscr{U}$ and $\mathscr{V}$ are diffeomorphic, if there is a differentiable map $F : \mathscr{U} \to \mathscr{V}$ which is one-to-one and onto. Moreover, the inverse mapping: $F^{-1} : \mathscr{U} \to \mathscr{V}$ is also differentiable. We also say that F is a diffeomorphism of $\mathscr{U}$ to $\mathscr{V}$. ♣

Theorem 1.5. (Inverse Function Theorem)

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping between Euclidean spaces of the same dimension. If $F_{*\mathbf{p}}$ is one-to-one at a point $\mathbf{p}$, there is an open set $\mathscr{U}$ containing $\mathbf{p}$ such that F restricted to $\mathscr{U}$ is a diffeomorphism of $\mathscr{U}$ onto an open set $\mathscr{V}$. ♡

External Link. *In the video here, I further elaborate the Chain Rule using the tangent map.*

Chapter 2 Frame Fields

Introduction

- *The dot product revisited*
- *The Frenet formulas*

2.1 Dot Product

We have discussed inner product in Chapter 0. The dot product is a special case of inner product. So we shall very quick go through it.

Definition 2.1

Let $\mathbf{p} = (p_1, p_2, p_3)$ *and* $\mathbf{q} = (q_1, q_2, q_3)$ *in* $\mathbb{R}^3$. *The dot product is defined by*

$$\mathbf{p} \cdot \mathbf{q} = p_1 q_1 + p_2 q_2 + p_3 q_3.$$

The norm of a point $\mathbf{p}$ *is defined by*

$$\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \mathbf{p}}.$$

The Euclidean distance from $\mathbf{p}$ *to* $\mathbf{q}$ *is the number*

$$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|.$$

Vectors are called orthogonal , if

$$\mathbf{p} \cdot \mathbf{q} = 0.$$

More generally, the angle θ *between vectors* $\mathbf{p}, \mathbf{q}$ *is defined by the equation*

$$\mathbf{p} \cdot \mathbf{q} = \|\mathbf{p}\| \cdot \|\mathbf{q}\| \cdot \cos\theta.$$

A vector $\mathbf{p}$ *is called a unit vector, if* $\|\mathbf{p}\| = 1$.

♣

Definition 2.2

A set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *of three mutually orthogonal unit vectors tangent to* $\mathbb{R}^3$ *at* $\mathbf{p}$ *is called a frame*[a] *at the point* $\mathbf{p}$.

[a] it is also called an orthonormal basis.

♣

Thus $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a frame if and only if

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1;$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0.$$

Using the Einstein Convention, we have $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Definition 2.3

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a frame at a point $\mathbf{p}$ of $\mathbb{R}^3$. The 3×3 matrix A whose rows are the Euclidean coordinates of these three vectors is called the attitude matrix of the frame. ♣

Explicitly, if

$$\mathbf{e}_1 = (a_{11}, a_{12}, a_{13})_{\mathbf{p}},$$
$$\mathbf{e}_2 = (a_{21}, a_{22}, a_{23})_{\mathbf{p}},$$
$$\mathbf{e}_3 = (a_{31}, a_{32}, a_{33})_{\mathbf{p}},$$

then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We shall prove that A is an *orthogonal* matrix: we have

$$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = a_{ik}a_{jk}.$$

The matrix expression of the above equation is

$$AA^T = I,$$

where A^T is the transpose of A.

Note *If $AA^T = I$, then $A^T = A^{-1}$. Therefore*

$$A^T A = A^{-1}A = I.$$

Note that this is not a trivial result. Take a 2×2 matrix, for example, Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $AA^T = I$ is equivalent to

$$a^2 + b^2 = 1, \qquad ac + bd = 0, \qquad c^2 + d^2 = 1,$$

and

$$A^T A = I$$

is equivalent to

$$a^2 + c^2 = 1, \qquad ab + cd = 0, \qquad b^2 + d^2 = 1.$$

Can you find an elementary proof of the above fact? Hint: we can prove the identity

$$(a^2 + c^2 - 1)^2 + (b^2 + d^2 - 1)^2 + 2(ab + cd)^2$$
$$= (a^2 + b^2 - 1)^2 + (c^2 + d^2 - 1)^2 + 2(ac + bd)^2.$$

2.2 Curves

Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a curve. It is well known that

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

is the velocity of the curve. The velocity is a vector field. Its norm, $\|\alpha'(t)\|$, is called the *speed* of the curve. The speed of a curve is a function along the curve.

In terms of the components, we can write

$$\nu = \|\alpha'\| = \left(\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2\right)^{1/2}.$$

The length of the curve, from $t = a$ to $t = b$, can be expressed by

$$\int_a^b \|\alpha'(t)\|\, dt.$$

Definition 2.4

A curve $\alpha : I \to \mathbb{R}^3$ is called regular, if the velocity vector field $\alpha'(t) \neq 0$ for any $t \in I$. This is equivalently to say that the speed function $\nu = \|\alpha'(t)\|$ is not zero at any point $t \in I$. If $\|\alpha'(t)\| = 1$, then the curve is called a unit speed curve. ♣

Definition 2.5

A reparametrization is a map $t : [c, d] \to [a, b]$ such that the function is one-to-one and onto, and that $t'(s) \neq 0$ for any $s \in [c, d]$. ♣

Proposition 2.1

Let $t(s)$ be a reparametrization. Then we have the following two cases:

(1) $t'(s) > 0$ *and* $t(c) = a, t(d) = b$;

(2) $t'(s) < 0$ *and* $t(c) = b, t(d) = a$. ♠

Proof. Since $t'(s) \neq 0$, we must have either $t'(s) > 0$ or $t'(s) < 0$. In the first case, $t(s)$ is monotonically increasing. Thus $t(c)$ must be minimal and hence equal to a, and $t(d)$ must be maximum, hence equal to b. This proves the first case. ■

Theorem 2.1

The length of a curve is an invariant. ♡

Proof. Let $t = t(s)$ be a reparametrization, that is $t : [c, d] \to [a, b]$ such that the function t is one-to-one and onto and we assume that $t'(s) \neq 0$. Let $s = s(t) : [a, b] \to [c, d]$ be the inverse function. Let $\beta(s) = \alpha(t(s))$. Then the length of the curve β, from c to d, is

$$\int_c^d \|\beta'(s)\| ds.$$

By the chain rule, $\beta'(s) = t'(s)\alpha'(t(s))$. We then have

$$\int_c^d \|\beta'(s)\| ds = \int_c^d |t'(s)| \cdot \|\alpha'(t(s))\| ds = \int_a^b \|\alpha'(t)\| \, dt.$$

■

Theorem 2.2

If α is a regular curve, then there is a parametrization β such that β has unit speed. ♡

Proof. Let $t = t(s)$ be a reparametrization and let $\beta(s) = \alpha(t(s))$. The requirement is that

$$1 = \|b'(s)\| = |t'(s)| \cdot \|\alpha'(t(s))\|$$

for any s. Thus in order to find the unit speed parametrization, we need to solve the differential equation

$$t'(s) = \frac{1}{\|\alpha'(t(s))\|}$$

with the initial value $t(0) = a$. This is a separable equation. We write $s'(t) = ds/dt$. Let $s = s(t)$ be the inverse function of $t(s)$. Then

$$s(t) = \int_a^t \|\alpha'(u)\| du$$

defines the unit speed reparametrization.

■

Note *Note that $s(a) = 0$, and $s(b)$ is the length of the curve from a to b.*

Example 2.1 Consider the helix

$$\alpha(t) = (a\cos t, a\sin t, bt).$$

The velocity of α is

$$\alpha'(t) = (-a\sin t, a\cos t, b).$$

The speed of α is

$$\|\alpha'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos t^2 + b^2} = \sqrt{a^2 + b^2}.$$

Therefore α has *constant speed* $c = \sqrt{a^2 + b^2}$.

Let

$$s(t) = \int_0^t c \, du = ct.$$

Then $t = s/c$. We thus have

$$\beta(s) = \alpha(s/c) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right)$$

is the unit speed reparametrization.

Definition 2.6

Let $Y = y_i U_i$ be a vector field along a curve α. Thus each function y_i can be expressed as a function of t via α. The differentiation of Y is simply differentiation on its Euclidean coordinate functions. ♣

Example 2.2 Let

$$Y(t) = t^2 U_1 - tU_3.$$

Then

$$Y'(t) = 2tU_1 - U_3.$$

In particular, we can define the *acceleration* $\alpha''(t)$ of the curve $\alpha(t)$.

Lemma 2.1

(1) *A curve α is constant if and only if its velocity is zero, $\alpha' = 0$.*

(2) *A non-constant curve α is a straight line if and only if its acceleration is zero, $\alpha'' = 0$.*

(3) *A vector field Y on a curve is parallel if and only if its derivative is zero, $Y' = 0$[a].*

[a]This is the definition of the parallelism rather than a statment.

♡

2.3 The Frenet Formulas

Let $\beta : I \to \mathbb{R}^3$ be a unit-speed curve. Let $T = \beta'(s)$ be the velocity vector field. Then we have $\|T\| = 1$. We consider $T' = \beta''(s)$. Since $\|T\| = 1$, we have $T \cdot T = 1$. Taking derivative[1] on both sides, we have $T' \cdot T + T \cdot T' = 0$. Thus $T \cdot T' = 0$, and T' is always orthogonal to T.

Definition 2.7

The curvature $\kappa(s) = \|T'(s)\| = \|\beta''(s)\|$.

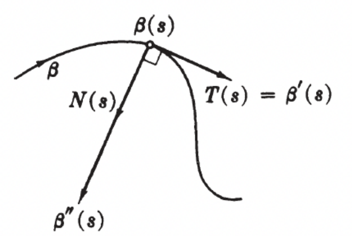


FIG. 2.7

♣

By the above definition, we know that $k(s) \geq 0$. In order to introduce the Frenet Formulas, we further assume that $\kappa > 0$.

[1]This technique will be used repeatedly throughout the rest of the book.

When $\kappa > 0$, we have $N = T'/\kappa$. By definition, we have $\|N\| = 1$. By the similar argument as above, we have $N \cdot N' = 0$.

Definition 2.8

Assume that $\kappa(s) > 0$. Define $N = T'/\kappa$, and $B = T \times N$. Then (T, N, B) is an orthonormal basis of the tangent space at point $\beta(s)$. We call (T, N, B) the Frenet frame field, or TNB frame field on β. The collection $\{T, N, B, \kappa, \tau\}$ is called the Frenet Apparatus.

♣

Remark If $\kappa \equiv 0$, then the curve is a straight line. On the other hand, if κ is nowhere zero, then we are able to define the Frenet frame field. It is beyond the scope of this book to discuss curves with vanishing curvature at isolated points.

By definition, we have $\|N\| = 1$. By the similar argument as above, we have $N \cdot N' = 0$. Thus

Definition 2.9

We can define

$$\tau = N' \cdot B,$$

where τ is called the torsion of the curve.

♣

Theorem 2.3. Frenet Formulas

If $\beta : I \to \mathbb{R}^3$ is a unit-speed curve with curvature $\kappa > 0$ and torsion τ, then we have the following system of ordinary differential equations:

$$\begin{aligned} T' &= \quad\quad \kappa N \\ N' &= -\kappa T \quad\quad + \tau B. \\ B' &= \quad\quad -\tau N \end{aligned}$$

In matrix notation, we have

$$(T, N, B)' = (T, N, B) \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}. \tag{2.1}$$

♡

Proof. The proof is an application of the orthogonality of the frame. The first equation follows by definition. Since $N \cdot N' = 0$, we can write

$$N' = aT + bB.$$

Thus $a = (N' - \tau B) \cdot T = N' \cdot T$. Since $N \cdot T = 0$, we have $N' \cdot T + N \cdot T' = 0$. Thus

$$N' \cdot T = -N \cdot T' = -N \cdot \kappa T = -\kappa.$$

By definition, we have

$$b = N' \cdot B = \tau.$$

This proves the second equation.

In order to obtain the third equation, we used the similar method. Write

$$B' = pT + qN + rB.$$

We have

$$p = B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0,$$

and

$$q = B' \cdot N = -B \cdot N' = -B \cdot (-\kappa T + \tau B) = -\tau,$$

and

$$r = B' \cdot B = 0.$$

This proves the third formula.

■

Example 2.3 We compute the Frenet frame T, N, B and the curvature and torsion functions of the unit-speed helix

$$\beta(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c}\right),$$

where $c = (a^2 + b^2)^{1/2}$ and $a > 0$. Now

$$T(s) = \beta'(s) = \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c}\right).$$

Hence

$$T'(s) = \left(-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0\right).$$

Thus

$$\kappa(s) = \|T'(s)\| = \frac{a}{c^2} = \frac{a}{a^2 + b^2} > 0.$$

Since $T' = \kappa N$, we get

$$N(s) = \left(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0\right).$$

Therefore, we have

$$B = T \times N = \left(\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c}\right).$$

The torsion is given by

$$\tau = N' \cdot B = \frac{b}{c^2}.$$

In what follows, we use Frenet formulas to study the properties of curves.

Definition 2.10

A plane curve in $\mathbb{R}^3$ *is a curve that lies in a single plane of* $\mathbb{R}^3$.

♣

Theorem 2.4

Let β *be a unit-speed curve in* $\mathbb{R}^3$ *with* $\kappa > 0$. *Then* β *is a plane curve if and only if* $\tau = 0$.

♡

Proof. If a curve $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s))$ lies in a plane, then there are constants a, b, c, d such that

$$a\beta_1 + b\beta_2 + c\beta_3 = d.$$

Using the vector notation, if $\mathbf{v} = (a, b, c)$, then $\mathbf{v} \cdot \beta = d$. As a result, we have

$$\mathbf{v} \cdot \beta'(s) = \mathbf{v} \cdot \beta''(s) = \mathbf{v} \cdot \beta'''(s) = 0.$$

Since $\beta' \perp \beta''$, they are linearly independent. Thus we can write β''' as a linear combination of β' and β''.

$$\beta''' = p\beta' + q\beta'',$$

where p, q are constants. Since

$$B = T \times N = \kappa^{-1}\beta' \times \beta'',$$

we have

$$\tau = N' \cdot B = (\kappa^{-1}\beta'')' \cdot B = 0.$$

On the other hand, if $\tau \equiv 0$, then $B' = 0$. Thus B is a constant vector. Let

$$f(s) = \beta(s) \cdot B - \beta(0) \cdot B.$$

Then $f' = 0$ and since $f(0) = 0$, we know that $f \equiv 0$. Thus $\beta(s)$ is on the plane

$$\mathbf{x} \cdot B - \beta(0) \cdot B = 0.$$

■

In the following, we shall use the Taylor's formula to study the curve β. We have

$$\beta(s) = \beta(0) + \beta'(0)s + \frac{1}{2}\beta''(0)s^2 + \frac{1}{6}\beta'''(0)s^3 + o(s^3).$$

Apparently, we have

$$\begin{aligned} &\beta'(0) = T(0), \\ &\beta''(0) = \kappa(0)N(0), \\ &\beta'''(0) = (\kappa N)' = \kappa'(0)N + \kappa(0)(-\kappa(0)T(0) + \tau(0)B(0)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\beta(s) - \beta(0) = sT(0) + \frac{1}{2}s^2\kappa(0)N(0) \\ &+ \frac{1}{6}s^3(\kappa'(0)N + \kappa(0)(-\kappa(0)T(0) + \tau(0)B(0))) + o(s^3). \end{aligned} \tag{2.2}$$

From the above formula, we know that the Frenet Apparatus completely determined the curve, at least for the first three terms in the Taylor's expansion.

Remark We have emphasized all along the distinction between a tangent vector and a point of $\mathbb{R}^3$. However, Euclidean space has, as we have seen, the remarkable property that given a point $\mathbf{p}$, there is a natural one-to-one correspondence between points (v_1, v_2, v_3) and tangent vectors $(v_1, v_2, v_3)_\mathbf{p}$ at $\mathbf{p}$. Thus one can transform points into tangent vectors (and vice versa) by means of this canonical isomorphism. In the next two sections particularly, it will often be convenient to switch quietly from one to the other without change of notation. Since corresponding objects have the same Euclidean coordinates, this switching can have no effect on scalar multiplication, addition, dot products, differentiation, or any other operation defined in terms of Euclidean coordinates.

So in (2.2), the left side $\beta(s) - \beta(0)$ is a vector on $\mathbb{R}^3$, and the right hand side, which is a linear combination of $T(0), N(0)$ and $B(0)$, is a vector in the tangent space of $\beta(0)$. We identify $T_{\beta(0)}\mathbb{R}^3$ with $\mathbb{R}^3$ so the equality makes sense.

2.4 Arbitrary-Speed Curves

It is a simple matter to adapt the results of the previous section to the study of a regular curve $\alpha : I \to \mathbb{R}^3$ that does not necessarily have unit speed. We merely transfer to α the Frenet apparatus of a unit-speed reparametrization $\bar{\alpha}$ of α. Explicitly, if s is an arc length function for α, then

$$\alpha(t) = \bar{\alpha}(s(t))$$

for all t. Or, we can write $\alpha(t) = \bar{\alpha}(s)$.

Assume that $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa} > 0, \bar{\tau}$ are the Frenet Apparatus with respect to the unit speed curve $\bar{\alpha}(s)$. Then we define

- *curvature* function: $\kappa = \bar{\kappa}(s)$,
- *torsion* function: $\tau = \bar{\tau}(s)$,
- *unit tangent* vector field: $T = \bar{T}(s)$,
- *principal* vector field: $N = \bar{N}(s)$,
- *binomial* vector field: $B = \bar{B}(s)$.

The speed of the curve is defined by $\nu = \|\alpha'(t)\|$. We can regard ν as a function of t but throught $t = t(s)$, it can be regarded as a function of s as well. Since $\alpha(t) = \bar{\alpha}(s)$, we have $\alpha'(t) = \bar{\alpha}'(s)\frac{ds}{dt}$. Thus we have

$$\left|\frac{ds}{dt}\right| = \|\alpha'(t)\| = \nu.$$

Lemma 2.2

Assume that $ds/dt > 0$[a]. If α is a regular curve in $\mathbb{R}^3$ with $\kappa > 0$, then we have

$$\begin{aligned} T' &= \kappa\nu N \\ N' &= -\kappa\nu T + \tau\nu B. \\ B' &= -\tau\nu N \end{aligned}$$

[a] We shall always assume this for the rest of the lecture notes. Thus we have $\nu = ds/dt$.

Proof. We shall use the Frenet formulas for unit speed curves. Since $T(t) = \bar{T}(s), N(t) = \bar{N}(s)$ and $B(t) = \bar{B}(s)$, we have $T'(t) = \bar{T}'(s)\nu, N'(t) = \bar{N}'(s)\nu$, and $B'(t) = \bar{B}(s)\nu$. The lemma then follows from Theorem 2.3. ∎

Note *There is a commonly used notation for the calculus that completely ignores change of parametrization. For example, the same letter would designate both a curve α and its unit-speed parametrization $\bar{\alpha}$, and similarly with the Frenet apparatus of these two curves. Differences in derivatives are handled by writing, say, dT/dt for $T'(t)$ and dT/ds for $\bar{T}'(s)$.*

Lemma 2.3

If α is a regular curve with speed function ν, then the velocity and acceleration of α are given by

$$\alpha' = \nu T, \quad \alpha'' = \frac{d\nu}{dt}T + \kappa\nu^2 N.$$

Proof. The proof is not difficult but the notations are confusing. By the context, $\alpha' = \alpha'(t)$ and $\alpha'' = \alpha''(t)$. We thus have

$$\alpha' = T\frac{ds}{dt} = T\,\nu,$$

and

$$\alpha'' = \frac{d\nu}{dt}T + \nu T' = \frac{d\nu}{dt}T + \kappa\nu^2 N.$$

∎

Remark The formula $\alpha' = \nu T$ is to be expected since α' and T are each tangent to the curve and T has a unit length, while $\|\alpha'\| = \nu$.

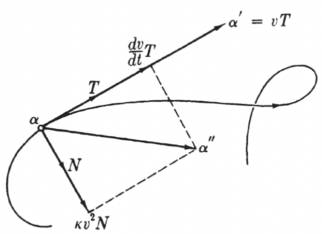


FIG. 2.15

The formula for acceleration is more interesting. By definition, α'' is the rate of change of the and in general both the length and the direction of α' are changing. The *tangential component* $\frac{d\nu}{dt}T$ of α'' measures the rate of change of the length of α' (that is, of the speed of α). The *normal component* $\kappa\nu^2 N$ measures the rate of change of the direction of α'. Newton's laws of motion show that these components may be experienced as forces. For example, in a car that is speeding up or slowing down on a straight road, the only force one feels is due to $\frac{d\nu}{dt}T$. If one takes an unbanked curve at speed ν, the resulting sideways force is due to $\kappa\nu^2 N$. Here κ measures how sharply the *road* turns; the effect of speed is given by ν^2, so 60 miles per hour is four times as unsettling as 30.

Note *Assume that we live in a* 1*-dimensional space defined by the above curve* α. *We then can only measure the tangential component of the acceleration* α''. *As a* 1*-dimensional creature, it is not possible for the creature to understand* κ*, the curvature of the curve. It would think it lives in a straight line.*

Theorem 2.5

Let α *be a regular curve in* $\mathbb{R}^3$. *Then*

$$T = \alpha'/\|\alpha'\|,$$

$$N = B \times T, \qquad \kappa = \|\alpha' \times \alpha''\|/\|\alpha'\|^3,$$

$$B = \alpha' \times \alpha''/\|\alpha' \times \alpha''\|, \qquad \tau = (\alpha' \times \alpha'') \cdot \alpha'''/\|\alpha' \times \alpha''\|^2.$$

♡

Proof. The proof is just a matter of applications of the chain rule and the Frenet formulas. But it contains several basic techniques in differential geometry.

First, we have $T = \alpha'/\|\alpha'\|$. Next, using the above Lemma 2.3, we have

$$\alpha' \times \alpha'' = \|\alpha'\|\, T \times (\nu' T + \kappa\nu^2 N) = \kappa\nu^3 B. \tag{2.3}$$

Taking the norm of the above equation, we prove the formula for κ is proved.

That $N = B \times T$ follows from the definition. From Lemma 2.3, we have

$$\alpha''' = (\nu' T)' + (\kappa\nu^2 N)'.$$

From the above, we know that the B components of α''' is $\kappa\tau\nu^3$. Thus we have

$$(\alpha' \times \alpha'') \cdot \alpha''' = \kappa^2\tau\nu^6.$$

Since $\|\alpha' \times \alpha''\|^2 = \kappa^2\nu^6$, the formula for τ is proved.

■

External Link. *As we know, the definition of the curvature is* $\kappa = \nu^{-1}\|T'(t)\|$. *Can we obtain the above formula by a straightforward computation. Yes, it is complicated, but such kind of computation contains useful techniques in differential geometry. See here for the method.*

Example 2.4 We compute the Frenet apparatus of the 3-curve

$$\alpha(t) = (3t - t^3, 3t^2, 3t + t^3).$$

We have

$$\alpha'(t) = 3\,(1-t^2, 2t, 1+t^2),$$
$$\alpha''(t) = 6\,(-t, 1, t),$$
$$\alpha'''(t) = 6\,(-1, 0, 1).$$

First, we have

$$\nu = \|\alpha'\| = \sqrt{\alpha' \cdot \alpha'} = 3\sqrt{2}(1+t^2).$$

We also have

$$\alpha' \times \alpha'' = 18 \begin{vmatrix} U_1 & U_2 & U_3 \\ 1-t^2 & 2t & 1+t^2 \\ -t & 1 & t \end{vmatrix} = 18\,(-1+t^2, -2t, 1+t^2).$$

Hence

$$\|\alpha' \times \alpha''\| = 18\sqrt{2}(1+t^2).$$

We compute

$$(\alpha' \times \alpha'') \cdot \alpha''' = 6 \cdot 18 \cdot 2.$$

It remains only to substitute this data into the formulas in Theorem 2.5 with N being computed by another cross product. The final results are

$$T = \frac{(1-t^2, 2t, 1+t^2)}{\sqrt{2}(1+t^2)},$$
$$N = \frac{(-2t, 1-t^2, 0)}{1+t^2},$$
$$B = \frac{(-1+t^2, -2t, 1+t^2)}{\sqrt{2}(1+t^2)},$$
$$\kappa = \tau = \frac{1}{3(1+t^2)^2}.$$

For the rest of the section, we shall do some applications of the above formulas.

Definition 2.11

The spherical image of a unit-speed curve $\beta(s)$ is the curve $\sigma(s) = T(s) = \beta'(s)$. ♣

Let σ be the spherical image of β. Then $\sigma' = \beta'' = \kappa N$, where κ is the curvature of β. That $\kappa > 0$ ensures that σ is a regular curve. In order to compute the curvature κ_σ of σ, we compute[2]

$$\sigma' = \beta'' = \kappa\, N,$$
$$\sigma'' = \beta''' = \kappa'\, N + \kappa\, N' = \kappa'\, N + \kappa(-\kappa\, T + \tau B).$$

[2]If $\kappa > 0$, then $\sigma'' \neq 0$. Thus the Frenet Apparatus always exists for σ.

Thus

$$\sigma' \times \sigma'' = \kappa^2(\kappa B + \tau T).$$

Using the formula in Theorem 2.5, we have

$$\kappa_\sigma = \frac{\|\sigma' \times \sigma''\|}{\nu^3} = (1 + (\tau/\kappa)^2)^{1/2} \geq 1.$$

Definition 2.12

A regular curve α in $\mathbb{R}^3$ is a cylindrical helix provided the unit tangent vector field T of α has constant angle θ with some fixed unit vector $\mathbf{u}$; that is, $T(t) \cdot \mathbf{u} = \cos\theta$ for all t. ♣

Theorem 2.6

A regular curve α with $\kappa > 0$ is a cylindrical helix if and only if the ratio τ/κ is constant. ♡

Proof. First assume that a unit speed curve is a cylindrical helix curve. Then there is a constant vector $\mathbf{u}$ such that $T \cdot \mathbf{u} = c$, a constant. Taking derivative on both sides, we get $\kappa N \cdot \mathbf{u} = 0$. Since $\kappa > 0$, we get $N \cdot \mathbf{u} = 0$. Therefore, if we write $\mathbf{u}$ as a linear combination of T, N, B, there would be no N component.

By the assumption, we would get

$$\mathbf{u} = \cos\theta\, T + \sin\theta\, B.$$

Taking derivative of the above equation again, we get

$$0 = \cos\theta\, \kappa\, N - \sin\theta\, \tau N.$$

We thus have $\kappa\cos\theta - \tau\sin\theta = 0$. Hence $\tau/\kappa = \cot\theta$ is a constant.

Conversely, if τ/κ is a constant, we write $\tau/\kappa = \cot\theta$. Let

$$\mathbf{u} = \cos\theta\, T + \sin\theta\, B.$$

Then

$$\mathbf{u}' = \cos\theta\, \kappa\, N - \sin\theta\, \tau\, N = 0.$$

Therefore $\mathbf{u}$ is a constant vector field. Obviously, $\|\mathbf{u}\| = 1$. Then $T \cdot \mathbf{u} = \cos\theta$ which means that the curve is a cylindrical helix. ■

Can we prove the above result by solving the Frenet differential equations? Yes, in the following, we give another proof of the fact that if $\tau/\kappa = c$ be a constant, then there must be a unit vector $\mathbf{u}$ such that $T \cdot \mathbf{u}$ is a constant.

We use (2.1) to obtain

$$(T,N,B)' = (T,N,B)\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} = \kappa\,(T,N,B)\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -c \\ 0 & c & 0 \end{bmatrix}.$$

Thus we have (See the remark below)

$$(T,N,B) = (T(0),N(0),B(0))e^{A\int_0^t \kappa(u)\,du},$$

where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -c \\ 0 & c & 0 \end{bmatrix}.$$

Since A is a skew-symmetric 3×3 matrix, it is singular[3]. Let $\mathbf{v}$ be a unit eigenvector of A with respect to the zero eigenvalue. Let $\mathbf{u}$ be the vector such that

$$\mathbf{u}\cdot(T(0),N(0),B(0)) = \mathbf{v}.$$

Then

$$T\cdot\mathbf{u} = \mathbf{u}\cdot T = \mathbf{u}\cdot(T(0),N(0),B(0))e^{A\int_0^t \kappa(u)\,du}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Note that $\mathbf{v}, A = 0$, we thus have

$$\mathbf{v}\,e^{A\int_0^t \kappa(u)\,du} = \mathbf{v}\sum_{k=0}^{\infty}\frac{1}{k!}A^k\left(\int_0^t \kappa(u)\,du\right)^k = \mathbf{v}.$$

Thus

$$T\cdot\mathbf{u} = \mathbf{v}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is a constant.

Remark Let $A(t)$ be a matrix-valued function. We consider the system of differential equations

$$A'(t) = A(t)B(t), \qquad A(0) = A_0,$$

where $B(t)$ is a matrix-valued function. Then in general, we don't have a formula for the (unique) solution. If for any t, t', we have

$$B(t)B(t') = B(t')B(t),$$

then we have the solution

$$A(t) = A_0\,e^{\int_0^t B(u)du}.$$

For further explanation, see here.

[3] One can prove this fact by a straightforward computation. Alternatively, we observe that for skew-symmetric matrices, all eigenvalues must be purely imaginary unless they are zero. Since all purely imaginary eigenvalues must be in pairs (conjugate eigenvalues), there must be at least one zero eigenvalue for odd dimensional matrices.

2.5 Covariant Derivatives

In this section, we shall define probably one of the most important concepts in differential geometry, or may be even one of the most important concepts in mathematics: *covariant derivatives* on vector fields.

Definition 2.13

Let W be a vector field on $\mathbb{R}^3$, and let $\mathbf{v}$ be a tangent vector to $\mathbb{R}^3$ at the point $\mathbf{p}$. Then the covariant derivative of W with respect to $\mathbf{v}$ is the tangent vector

$$\nabla_{\mathbf{v}} W = W(\mathbf{p} + t\mathbf{v})'(0)$$

at the point $\mathbf{p}$. ♣

On surface, the covariant derivative is just another version of directional derivative, especially we have already defined such kind of derivative along a curve before. But we shall systematically use this kind of derivative under the frame of differential operators.

Example 2.5 Let $W = x^2 U_1 + yzU_3$, and let

$$\mathbf{v} = (-1, 0, 2),$$

at $\mathbf{p} = (2, 1, 0)$. Then

$$\mathbf{p} + t\mathbf{v} = (2 - t, 1, 2t).$$

So

$$W(\mathbf{p} + t\mathbf{v}) = (2 - t)^2 U_1 + 2tU_3.$$

Hence

$$\nabla_{\mathbf{v}} W = W(\mathbf{p} + t\mathbf{v})'(0) = -4\,U_1(\mathbf{p}) + 2\,U_3(\mathbf{p}).$$

Lemma 2.4

If $W = w_i U_i$ is a vector field on $\mathbb{R}^3$, and $\mathbf{v}$ is a tangent vector at $\mathbf{p}$, then

$$\nabla_{\mathbf{v}} W = \mathbf{v}[w_i] U_i(\mathbf{p}).$$

♡

Proof. We have[a]

$$W(\mathbf{p} + t\mathbf{v}) = w_i(\mathbf{p} + t\mathbf{v}) U_i(\mathbf{p} + t\mathbf{v}).$$

Differentiation of the above at $t = 0$, we obtain

$$\nabla_{\mathbf{v}} W = W(\mathbf{p} + t\mathbf{v})'(0) = \mathbf{v}[w_i] U_i(\mathbf{p}).$$

■

[a] This equation is from the book which is, of course, correct. But in all previous formulas, we use U_i instead of $U_i(\mathbf{p} + t\mathbf{v})$.

In short, to apply $\nabla_{\mathbf{v}}$ to a vector field, apply $\mathbf{v}$ to its Euclidean coordinates.

Theorem 2.7

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be tangent vectors* $\mathbb{R}^3$ *at* $\mathbf{p}$*, and let* Y *and* Z *be vector fields on* $\mathbb{R}^3$*. Then for numbers* a, b *and function* f

(1) $\nabla_{a\mathbf{v}+b\mathbf{w}}Y = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{w}}Y,$

(2) $\nabla_{\mathbf{v}}(aY + bZ) = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{v}}Z,$

(3) $\nabla_{\mathbf{v}}(fY) = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_{\mathbf{v}}Y,$

(4) $\mathbf{v}[Y \cdot Z] = \nabla_{\mathbf{v}}Y \cdot Z(\mathbf{p}) + Y(\mathbf{p}) \cdot \nabla_{\mathbf{v}}Z.$

♡

Proof. The proof is straightforward. For example, to prove (3), let $Y = y_iU_i$ and then $fY = (fy_i)U_i$. Thus

$$\nabla_{\mathbf{v}}(fY) = \mathbf{v}[fy_i]U_i = \mathbf{v}[f]y_iU_i + f\mathbf{v}[y_i]U_i = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_{\mathbf{v}}Y.$$

■

If we regard the set of vector fields as a module over the algebra of smooth functions, then the operator $\nabla_{\mathbf{v}}$ is linear with respect to the addition and scalar multiplication. It is a *derivative* on the multiplication of smooth functions to vector fields.

Corollary 2.1

Let V, W, Y *and* Z *be vector fields on* $\mathbb{R}^3$*;* f, g *be smooth functions. Then*

(1) $\nabla_{fV+gW}Y = f\nabla_V Y + g\nabla_W Y,$

(2) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z,$

(3) $\nabla_V(fY) = V[f]Y + f\nabla_V Y,$

(4) $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z.$

♡

2.6 Frame Fields

When the Frenet formulas were discovered (by Frenet in 1847, and independently by Serret in 1851), the theory of surfaces in $\mathbb{R}^3$ was already a richly developed branch of geometry. The success of the Frenet approach to curves led Darboux (around 1880) to adapt this "method of moving frames" to the study of surfaces. Then it was Cartan who brought the method to full generality. His essential idea was very simple: To each point of the object under study (a curve, a surface, Euclidean space itself, . . .) assign a frame; then using orthonormal expansion express the rate of change of the frame in terms of the frame itself. This, of course, is just what the Frenet formulas do in the case of a curve.

In the next three sections we shall carry out this scheme for the Euclidean space $\mathbb{R}^3$. We shall see that geometry of curves and surfaces in $\mathbb{R}^3$ is not merely an analogue, but actually a corollary, of these basic results.

Definition 2.14

Vector fields E_1, E_2, E_3 *on* $\mathbb{R}^3$ *constitute a frame field on* $\mathbb{R}^3$ *provided*

$$E_i \cdot E_j = \delta_{ij}.$$

♣

Apparently, U_1, U_2, U_3 is a frame field. We call such a frame field *Euclidean frame field*. In the following, we shall introduce two more important frame fields.

Example 2.6 (*The cylindrical frame field*) Let r, θ, z be the usual cylindrical coordinate functions on $\mathbb{R}^3$. We shall pick a unit vector field in the direction in which each coordinate increases (when the other two are held constant). For r, this is evidently

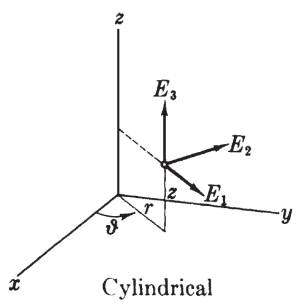
$$E_1 = \cos\theta\, U_1 + \sin\theta\, U_2.$$

Then

$$E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2$$

points in the direction of increasing θ as in Fig. 2.19. Finally, the direction of increase of z is, of course, straight up, so

$$E_3 = U_3.$$



Cylindrical

FIG. 2.19

Remark I don't think in the above example, deducing the cylindrical frame field from the cylindrical coordinates by the picture, is mathematically rigid, or *even correct*. In fact, as long as the above $\{E_1, E_2, E_3\}$ is an orthonormal basis at each point, it defines a frame field. We can *call* it the cylindrical frame field. There is no proof needed here.

A better way to show the relationship between the cylindrical coordinates and the cylindrical frame field is to use the chain rule. We have $x = r\cos\theta, y = r\sin\theta, z = z$. Then by the chain rule, for any function f, we have

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\cdot\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\cdot\frac{\partial f}{\partial y} + \frac{\partial z}{\partial r}\cdot\frac{\partial f}{\partial z}.$$

In terms of vector notation, this implies that

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}U_1 + \frac{\partial y}{\partial r}U_2 + \frac{\partial z}{\partial r}U_3 = \cos\theta\, U_1 + \sin\theta\, U_2.$$

Using the same method, we have

$$\frac{\partial}{\partial z} = U_3.$$

However,

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}U_1 + \frac{\partial y}{\partial\theta}U_2 + \frac{\partial z}{\partial\theta}U_3 = -r\sin\theta\, U_1 + r\cos\theta\, U_2$$

which is not E_2 but rE_2.

In general, if $\{x_1, x_2, x_3\}$ are another set of coordinate functions, then the vector fields

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$$

do not define a frame field (they may not be orthonormal).

Example 2.7 (*The spherical frame field*) Now we define the spherical frame field $\{F_1, F_2, F_3\}$. Let ρ, θ, φ be the spherical coordinates. We then have

$$x = \rho\cos\varphi\cos\theta,$$
$$y = \rho\cos\varphi\sin\theta,$$
$$z = \rho\sin\varphi.$$

The spherical frame field is defined by

$$F_1 = \cos\varphi\,(\cos\theta\, U_1 + \sin\theta\, U_2) + \sin\varphi\, U_3$$
$$F_2 = -\sin\theta\, U_1 + \cos\theta\, U_2,$$
$$F_3 = -\sin\varphi\,(\cos\theta\, U_1 + \sin\theta\, U_2) + \cos\varphi\, U_3.$$

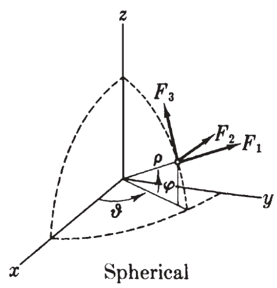


FIG. 2.20

By a straightforward computation, we have

$$\frac{\partial}{\partial\rho} = E_1,$$
$$\frac{\partial}{\partial\theta} = -\rho\cos\varphi\sin\theta\, U_1 + \rho\cos\varphi\cos\theta\, U_2 = \rho\cos\varphi\, F_2,$$
$$\frac{\partial}{\partial\varphi} = -\rho\sin\varphi\cos\theta\, U_1 - \rho\sin\varphi\sin\theta\, U_2 + \rho\cos\varphi\, U_3 = \rho\, F_3.$$

Lemma 2.5

Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$.

(1). If V is a vector field on $\mathbb{R}^3$, then $V = f_i E_i$, where the functions $f_i = V \cdot E_i$ are called coordinate functions of V with respect to $\{E_1, E_2, E_3\}$.

(2). If $V = f_i E_i$ and $W = g_i E_i$, then $V \cdot W = f_i g_i$. In particular, $\|V\| = (\sum f_i^2)^{1/2}$. ♡

2.7 Connection forms

Let $\{E_1, E_2, E_3\}$ be a frame field, and let $\mathbf{v}$ be a vector at $\mathbf{p} \in \mathbb{R}^3$. Let $\mathbf{p} \in \mathbb{R}^3$. Then we are able to write

$$\nabla_{\mathbf{v}} E_1 = c_{11}E_1(\mathbf{p}) + c_{12}E_2(\mathbf{p}) + c_{13}E_3(\mathbf{p}),$$
$$\nabla_{\mathbf{v}} E_2 = c_{21}E_1(\mathbf{p}) + c_{22}E_2(\mathbf{p}) + c_{23}E_3(\mathbf{p}),$$
$$\nabla_{\mathbf{v}} E_3 = c_{31}E_1(\mathbf{p}) + c_{32}E_2(\mathbf{p}) + c_{33}E_3(\mathbf{p}).$$

Using the Einstein Convention, we have

$$\nabla_{\mathbf{v}} E_i = c_{ij}E_j(\mathbf{p}).$$

Since $\{E_1, E_2, E_3\}$ is a frame, we then have[4]

$$c_{ij} = \nabla_{\mathbf{v}} E_i \cdot E_j(\mathbf{p}).$$

We then can define a one form ω_{ij} such that

$$\omega_{ij} = c_{ij} = \nabla_{\mathbf{v}} E_i \cdot E_j(\mathbf{p}).$$

Lemma 2.6

Using the above notations, then ω_{ij} are 1-forms satisfying

$$\omega_{ij} = -\omega_{ji}.$$

These 1-forms are called connection forms of the frame field $\{E_1, E_2, E_3\}$. ♡

Proof. In order to prove ω_{ij} is a 1-form, we just need to prove that

$$\omega_{ij}(a\mathbf{v} + b\mathbf{w}) = a\,\omega_{ij}(\mathbf{v}) + b\,\omega_{ij}(\mathbf{w}),$$

where $\mathbf{v}, \mathbf{w}$ are vectors and a, b are real numbers. But this follows from

$$\nabla_{a\mathbf{v}+b\mathbf{w}} E_i = a\,\nabla_{\mathbf{v}} E_i + b\,\nabla_{\mathbf{w}} E_i.$$

To prove $\omega_{ij} = -\omega_{ji}$, we observe that since $E_i \cdot E_j = \delta_{ij}$. Then

$$0 = \mathbf{v}[E_i \cdot E_j] = \nabla_{\mathbf{v}} E_i \cdot E_j + E_i \cdot \nabla_{\mathbf{v}} E_j = \omega_{ij}(\mathbf{v}) + \omega_{ji}(\mathbf{v}).$$

■

Theorem 2.8

Let ω_{ij} be the connection 1-froms of a frame field $\{E_1, E_2, E_3\}$ on $\mathbb{R}^3$. Then for any vector field V on $\mathbb{R}^3$, we have

$$\nabla_V E_i = \omega_{ij}(V)E_j.$$

♡

[4] We know that $1 \leq i, j \leq 3$. But this is implied by the Einstein convention so can be omitted.

By $\omega_{ij} = -\omega_{ji}$, we have $\omega_{ii} = 0$. We write

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}.$$

If we write

$$E = (E_1, E_2, E_3),$$

then in matrix notations, we have

$$\nabla_{\mathbf{v}} E = E \cdot \omega^T.$$

This can be used to compare with the Frenet formula

$$(T, N, B)' = (T, N, B) \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$

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