The Geometry of Triangles

UCI Math

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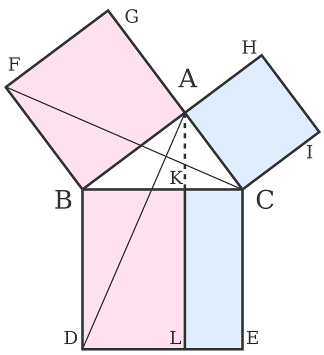

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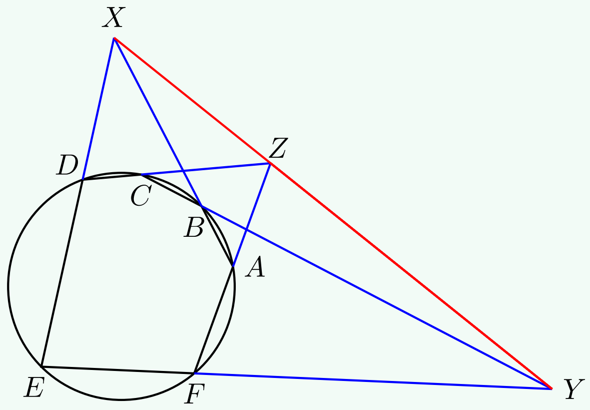
Problem 1. Pascal Theorem

The hexagon $ABCDEF$ is inscribed to the circle. Assume that AB, CD intersects at X; BC, EF intersects at Y; and CD, FA intersects at Z. Then X, Y, Z are collinear.



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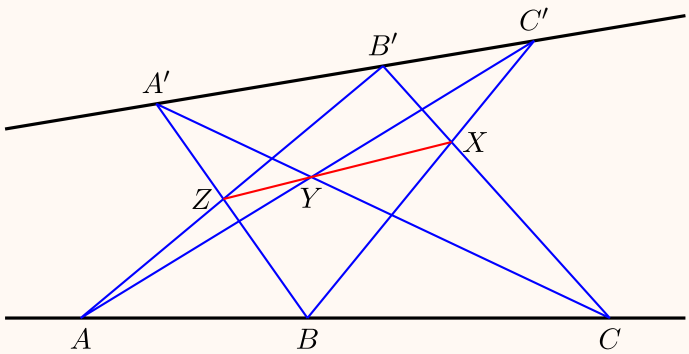
Remark The above is called Pascal Theorem, which was discovered by the French mathematician *Blaise Pascal*. when he was 16 years old. The theorem can be generalized to the case to the case of conic section (see Wikipedia here). When the conic section is degenerated to two lines, it is also called the *Pappus Hexagon Theorem* below – some people believed that Euclid knew this theorem before Pappus.

Theorem. (Pappus Theorem)

In the following picture, the Hexagon $BC'AB'CA'$ is inscribed on the two black lines. Assume that BC', $B'C$ intersect at X; CA', $A'C$ intersect at Y, and AB', $B'A$ intersect at Z. Then X, Y, Z are collinear.



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Proof. We use analytic geometry to prove the Pascal Theorem.

We assume the circle is the unit circle. Let the equations for

$$AB, BC, CD, DE, EF, FA$$

be $\ell_1, \ell_2, \cdots, \ell_6$. These functions ℓ_j are linear functions. As a result, we consider two cubic polynomials $\ell_1\ell_3\ell_5$ and $\ell_2\ell_4\ell_6$. Obviously, these two polynomials pass the nine points $A, B, C, D, E, F, X, Y, Z$.

We choose a general point P in the circle. Choose a number λ such that

$$(\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6)(P) = 0.$$

Here is a fundamental question: in general, if a cubic curve doesn't vanishing identically on the unit circle, then what is the maximum number of intersections? The answer is 6, and we shall prove it.

We can use complex numbers to write any cubic polynomials as

$$f(z) = Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3 + Ez^2 + Fz\bar{z} + G\bar{z}^2 + Hz + I\bar{z} + J = 0.$$

Let $z = e^{i\theta}$ be a point on the unit disk, then $\bar{z} = 1/z$. If we multiply the above equation by z^3 on both sides, we get a degree 6 polynomial of single variables. In general, such a 6-degree polynomial has at most 6 roots. Since $f(z)$ vanishes on severn points A, B, C, D, E, F, P on the unit circle, it must be vanishing identically on the circle. As a result, we can factorize it as

$$f(z) = (|z|^2 - 1)\ell(z),$$

where, by the degree consideration, $\ell(z)$ must be linear. Since ℓ passes X, Y, Z, we conclude that X, Y, Z are collinear.

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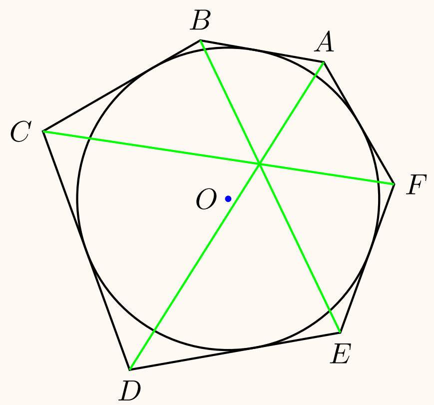
Theorem. (Brianchon Theorem)

The Hexagon $ABCDEF$ is circumscribed on a circle. Then AD, BE, and CF are concurrent.



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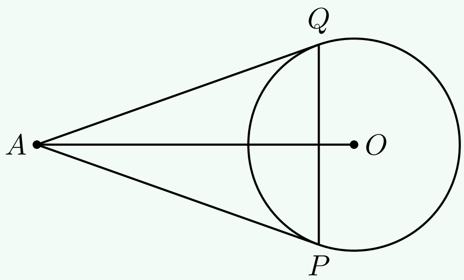
Remark Pascal Theorem and Brainchon Theorem are two famous "dual" theorems. We can prove it using Pascal Theorem.

Definition. (Pole and Polar)

Let O be the unit circle. The the pair (A, PQ) is called the pair of pole and polar, where A is the pole, and PQ is the polar.

Let (x_0, y_0) be the coordinates of A. Then the equation of PQ is

$$x_0x + y_0y - 1 = 0.$$



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Proof. Let the coordinates of A_i be (x_i, y_i) for $1 \leq i \leq 6$. Then the equations for B_6B_1 is

$$\ell_1(x, y) = x_1x + y_1y - 1.$$

Similarly, the equations for B_iB_{i+1} for $1 \leq i \leq 5$ are

$$\ell_i(x, y) = x_ix + y_iy - 1.$$

Using the Pascal Theorem, there is a number λ such that

$$\ell_1\ell_3\ell_5 + \lambda\ell_2\ell_4\ell_6 = C(x^2 + y^2 - 1)(px + qy - 1),$$

where C is a constant. We claim (p, q) is on the lines A_1A_4, A_2A_5, and A_3A_6.

In order to prove this, let $P = (p_1, q_1)$ be the intersection of B_1B_6 and B_3B_4. Then we have

$$x_1p_1 + y_1q_1 - 1 = 0, \qquad x_4p_1 + y_4q_1 - 1 = 0.$$

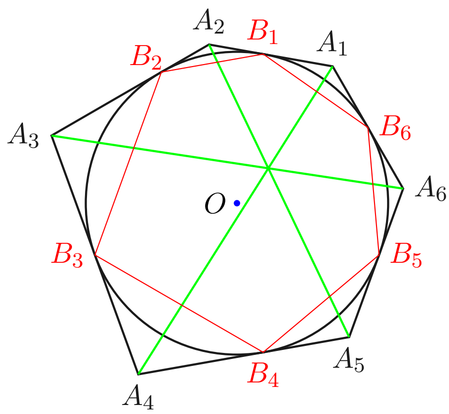
Moreover, we have

$$pp_1 + qq_1 - 1 = 0.$$

Thus the three points A_1, A_4 and (p, q) are on the line

$$p_1x + q_1y - 1 = 0.$$

This completes the proof.



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