Ukrainian Mathematical Journal, Vol. 51, No. 2, 1999

IMBEDDING OF THE IMAGES OF OPERATORS AND REFLEXIVITY OF BANACH SPACES

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UDC 517.983.23

We establish a criterion of reflexivity for a separable Banach space in terms of the relation between the imbedding of the images, factorization, and majorization of operators acting in this space.

Let A and B be linear continuous operators acting in a Banach space X. We are concerned with the relation between the following conditions.

(*i*) $B = AC$ for some linear continuous operator $C: X \to X$;

(*ii*) $\|B^*x^*\| \le K\|A^*x^*\|$ for some $K \ge 0$ and every $x^* \in X^*$;

(*iii*) $\operatorname{Im} B \subset \operatorname{Im} A$, where $\operatorname{Im}(A) = A(X)$.

If X is a Hilbert space, then conditions (i)–(iii) are equivalent. Moreover, the implications (i) ⇒ (ii), (i) ⇒ (iii), and (iii) ⇒ (ii) hold in an arbitrary Banach space X. However, as shown by Douglas, the implication (iii) ⇒ (i) may not always hold [2]. In [2], the question was posed as to whether (ii) ⇒ (iii) always holds and it was reported without proof in the addendum that Bouldin found a counterexample for which this implication is not true. Here, we give a characterization of separable Banach spaces for which condition (ii) implies condition (iii).

Theorem 1. *If X is a reflexive Banach space, then (ii) ⇔ (iii). If X is a separable nonreflexive Banach space, then there exist nuclear operators A and B acting in X for which condition (ii) is satisfied but conditions (i) and (iii) do not.*

First of all, we note that condition (ii) is equivalent to the imbedding $\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}$. Indeed, as is shown in [2], for linear operators D and E acting in a Banach space Y, the condition

$$\|Ey\| \le K\|Dy\| \quad \text{for some } K \text{ and all } y \in Y$$

is equivalent to the imbedding $\operatorname{Im} E^* \subset \operatorname{Im} D^*$. It remains to set $Y = X^*$, $D = A^*$, and $E = B^*$.

Lemma 1. *Let X be a nonreflexive separable Banach space. Then there exist $w^{**} \in X^{**}$ and a biorthogonal sequence $\{x_n, x_n^*\}_{n=1}^{\infty} \subset X \times X^*$ such that the infinite system of linear equations*

$$w^{**}(x_n^*) = x_n^*(x), \quad n = 1, 2, \ldots,$$

is not satisfied for any $x \in X$.

Proof. Let us imbed the space X canonically into X^{**} [3, p. 199] and consider X as a subspace of X^{**}. Since $X \ne X^{**}$, by virtue of the Riesz lemma [3, p. 73] there exists $w^{**} \in X^{**}$, $\|w^{**}\| = 1$, such that

Kharkov University, Kharkov. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 51, No. 2, pp. 260–262, February, 1999. Original article submitted November 4, 1996; revision submitted June 9, 1997.

0041–5995/99/5102–0293 $22.00

$$\| w^{**} - x \| \geq 2/3 \quad \text{for every} \quad x \in X.$$

Then, by the definition of the norm of a functional X^{**}, for every $x \in X$, there exists $y^* \in X^*$, $\| y^* \| = 1$, such that

$$| w^{**}(y^*) - y^*(x) | \geq 1/2.$$

Thus, if $\{y_n\}_{n=1}^{\infty}$ is an everywhere dense subset of X, there exist functionals $\{y_n^*\}_{n=1}^{\infty} \subset X^*$, $\|y_n^*\| = 1$, such that

$$\left| w^{**}\left(y_n^*\right) - y_n^*(y_n) \right| \geq 1/2, \quad n = 1, 2, \ldots .$$

For an arbitrary $x \in X$, there exists a number n such that $\| x - y_n \| \leq 1/2$. Therefore,

$$\left| w^{**}\left(y_n^*\right) - y_n^*(x) \right| \geq 1/4.$$

In other words,

$$w^{**} - x \notin \left(\left[y_n^*\right]_{n=1}^{\infty}\right)^{\perp} \quad \text{for every} \quad x \in X. \tag{1}$$

(Here, $[x_n]_{n=1}^{\infty}$ denotes the closure of a linear hull of vectors x_n, $n = 1, 2, \ldots$).

Let us now orthogonalize $\{ x_n \}$ by using the method of Markushevich [4, p. 43]. We have that $Y = \left[y_n^*\right]_{n=1}^{\infty}$ is a separable subspace of X^* and $\{y_n\}_{n=1}^{\infty} \subset X^{**}$ is a total set over X^*. Therefore, there exists a biorthogonal system $\left\{x_n^*, x_n\right\}_{n=1}^{\infty} \subset X^* \times X^{**}$ such that

$$\{x_n\}_{n=1}^{\infty} \subset [y_n]_{n=1}^{\infty} = X$$

and

$$\left[x_n^*\right]_{n=1}^{\infty} = Y.$$

By virtue of (1),

$$w^{**} - x \notin \left(\left[x_n^*\right]_{n=1}^{\infty}\right)^{\perp} \quad \text{for every} \quad x \in X.$$

Lemma 1 is proved.

Proof of Theorem 1. If X is reflexive, then $A = A^{**}$, $B = B^{**}$ and, therefore, condition (iii) is equivalent to the imbedding $\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}$. In view of the remark made above, the last is equivalent to condition (ii).

Let now X be nonreflexive and separable. Since condition (i) always implies condition (iii), it suffices to find some operators A and B such that

$$\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}, \ \operatorname{Im} B \not\subset \operatorname{Im} A.$$

We apply now Lemma 1. Let $\{y_n\}_{n=1}^{\infty}$ be an ω-linearly independent sequence in X (i.e., the equality

$$\sum_{n=1}^{\infty} a_n y_n = 0$$

implies $a_n = 0$ for all n) such that the series $\sum_{n=1}^{\infty} \|y_n\| \|x_n^*\|$ converges. Then the nuclear operators

$$A = \sum_{n=1}^{\infty} x_n^* \otimes y_n, \quad B = \sum_{n=1}^{\infty} x_n^* \otimes u_n$$

are well defined. Here,

$$u_n = \begin{cases} \sum_{i=1}^{\infty} w^{**}(x_i^*) y_i & \text{for } n = 1; \\ y_n & \text{for } n = 2, 3, \ldots . \end{cases}$$

Let us show that $\operatorname{Im} B \not\subset \operatorname{Im} A$. Indeed, we have $Bx_1 = u_1 \notin \operatorname{Im} A$: If $Au = u_1$ for some $u \in X$, i.e.,

$$\sum_{n=1}^{\infty} x_n^*(u) y_n = u_1 = \sum_{n=1}^{\infty} w^{**}(x_n) y_n,$$

then, by virtue of the ω-linear independence of $\{y_n\}_{n=1}^{\infty}$, it would follow that $x_n^*(u) = w^{**}(x_n)$, $n = 1, 2, \ldots$, which contradicts the assertion of Lemma 1.

Let us show that $\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}$. We have

$$A^{**} = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

$$B^{**} = \sum_{n=1}^{\infty} x_n^* \otimes u_n = D_1 + D_2,$$

where

$$D_1 = x_n^* \otimes \left(\sum_{n=1}^{\infty} w^{**}(x_n^*) y_n \right), \quad D_2 = \sum_{n=1}^{\infty} x_n^* \otimes y_n.$$

(Here, we interpret the functionals x_n^* as elements of X^{***}.)

We fix an arbitrary $x^{**} \in X^{**}$. Then, for the vector $u_1^{**} = x^{**}(x_1^*) w^{**}$, we have $Au_1^{**} = D_1 x^{**}$. Therefore, $\operatorname{Im} D_1 \subset \operatorname{Im} A$. Further, let P be a projector in $[x_n^*]_{n=1}^{\infty}$ onto $[x_n^*]_{n=2}^{\infty}$ along x_1^*. Consider the vector $\hat{u}_2^{**} = P^* \hat{x}^{**}$, where $\hat{x}^{**}$ is the restriction of the functional x^{**} to the subspace $[x_n^*]_{n=1}^{\infty}$. Thus, if $u_2^{**} \in X^{**}$ is an arbitrary extension of the functional $\hat{u}_2^{**}$, then

$$A^{**} u_2^{**} = \sum_{n=1}^{\infty} u_2^{**}(x_n^*) y_n = \sum_{n=1}^{\infty} P^* \hat{x}^{**}(x_n^*) y_n = \sum_{n=2}^{\infty} \hat{x}^{**}(x_n^*) y_n = D_2 x^{**}.$$

Therefore, $\operatorname{Im} D_2 \subset \operatorname{Im} A^{**}$.

Consequently, $\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}$, and Theorem 1 is proved.

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