

INTEGER CELLS IN CONVEX SETS

R. VERSHYNIN

ABSTRACT. Every convex body K in \mathbb{R}^n has a coordinate projection PK that contains at least $\text{vol}(\frac{1}{6}K)$ cells of the integer lattice $P\mathbb{Z}^n$, provided this volume is at least one. Our proof of this counterpart of Minkowski's theorem is based on an extension of the combinatorial density theorem of Sauer, Shelah and Vapnik-Chervonenkis to \mathbb{Z}^n . This leads to a new approach to sections of convex bodies. In particular, fundamental results of the asymptotic convex geometry such as the Volume Ratio Theorem and Milman's duality of the diameters admit natural versions for coordinate sections.

1. INTRODUCTION

Minkowski's Theorem, a central result in the geometry of numbers, states that if K is a convex and symmetric set in \mathbb{R}^n , then $\text{vol}(K) > 2^n$ implies that K contains a nonzero integer point. More generally, K contains at least $\text{vol}(\frac{1}{2}K)$ integer points. The main result of the present paper is a similar estimate on the number of *integer cells*, the unit cells of the integer lattice \mathbb{Z}^n , contained in a convex body.

Clearly, the largeness of the volume of K does not imply the existence of any integer cells in K ; a thin horizontal pancake is an example. The obstacle in the pancake K is caused by only one coordinate in which K is flat; after eliminating it (by projecting K onto the remaining ones) the projection PK will have many integer cells of the lattice $P\mathbb{Z}^n$. This observation turns out to be a general principle.

Theorem 1.1. *Let K be a convex set in \mathbb{R}^n . Then there exists a coordinate projection P such that PK contains at least $\text{vol}(\frac{1}{6}K)$ cells of the integer lattice $P\mathbb{Z}^n$, provided this volume is at least one.*

(A coordinate projection is the orthogonal projection in \mathbb{R}^n onto \mathbb{R}^I for some nonempty subset $I \subset \{1, \dots, n\}$.)

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Combinatorics: Sauer-Shelah-type results. Theorem 1.1 is a consequence of an extension to \mathbb{Z}^n of the famous result due to Vapnik-Chervonenkis, Sauer, Perles and Shelah, commonly known as Sauer-Shelah lemma, see e.g. [6, §17].

Sauer-Shelah Lemma. *If $A \subset \{0, 1\}^n$ has cardinality $\#A > \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}$, then there exists a coordinate projection P of rank larger than d and such that $PA = P\{0, 1\}^n$.*

This result is used in a variety of areas ranging from logics to theoretical computer science to functional analysis [18]. In order to bring Sauer-Shelah Lemma to geometry, we will need first to generalize it to sets $A \subset \mathbb{Z}^n$. An *integer box* is a subset of \mathbb{Z}^I of the form $\prod_{i \in I} \{a_i, b_i\}$ with $a_i \neq b_i$.

Theorem 1.2. *If $A \subset \mathbb{Z}^n$, then*

$$\#A \leq 1 + \sum_P \#(\text{integer boxes in } PA),$$

where the sum is over all coordinate projections P .

If $A \subset \{0, 1\}^n$, then every PA in the sum above may contain only one integer box $P\{0, 1\}^n$ if any, hence

$$(1) \quad \#A \leq 1 + \#(P \text{ for which } PA = P\{0, 1\}^n).$$

Estimate (1) is due to A.Pajor [27]. Since the right hand side of (1) is bounded by $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}$, where d is the maximal rank of P for which $PA = P\{0, 1\}^n$, (1) immediately implies Sauer-Shelah Lemma.

In a similar way, Theorem 1.2 implies a recent generalization of Sauer-Shelah lemma in terms of Natarajan dimension, due to Haussler and Long [12]. In their result, A has to be bounded by some paralelepiped; we do not impose any boundedness restrictions (see Corollary 2.6).

Most importantly, Theorem 1.2 admits a version for integer *cells* instead of integer boxes. If $A \subset \mathbb{R}^n$ is convex, then

$$\#A \leq 1 + \sum_P \#(\text{integer cells in } PA).$$

This quickly leads to Theorem 1.1. This version also implies a generalization of Sauer-Shelah lemma from [12] in terms of the *combinatorial dimension*, which is an important concept originated in the statistical learning theory and which became widely useful in many areas, see [1], [7], [39], [20]. These results will be discussed in detail in Section 2. The proof of Theorem 1.2 relies on the combinatorics developed in [20] and [30].

Convex geometry: coordinate sections of convex bodies. Theorem 1.2 leads to a new approach to coordinate sections of convex bodies.

The problem of finding nice coordinate sections of a symmetric convex body K in \mathbb{R}^n has been extensively studied in geometric functional analysis. It is connected in particular with important applications in harmonic analysis, where the system of characters defines a natural coordinate structure. The Λ_p -problem, which was solved by J. Bourgain [4], is an exemplary problem on finding nice coordinate sections, as explained by an alternative and more general solution (via the majorizing measures) given by M. Talagrand [38]. It is generally extremely difficult to find a nice coordinate section even when the existence of nice generic sections (usually randomly chosen from the Grassmanian) is well known, see e.g. [39], [20], [30].

The method of the present paper allows one to prove natural versions of a few classical results for coordinate sections. Since the number of integer cells in a set K is bounded by its volume, we have in Theorem 1.1 that

$$(2) \quad PK \text{ contains an integer cell and } |PK| \geq |\frac{1}{6}K|.$$

(we write $|PK| = \text{vol}(PK)$ for the volume in $P\mathbb{R}^n$). This often enables one to conclude *a posteriori* that P has large rank, as (2) typically fails for all projections of small ranks.

If K is symmetric and an integer $m < n$ is fixed, then using (2) for $a^{-1}K$ with an appropriate $a > 0$, we obtain $a^{-m}|PK| \geq a^{-n}|\frac{1}{6}K|$ for some coordinate projection P of rank m . Moreover, $P(a^{-1}K)$ contains a unit coordinate cube, so solving for a we conclude that

$$(3) \quad PK \text{ contains a coordinate cube of side } \left(\frac{|cK|}{|PK|} \right)^{\frac{1}{n-m}}.$$

where C, c, c_1, \dots denote positive absolute constants (here $c = 1/6$).

This leads to a “coordinate” version of the classical Volume Ratio Theorem. This theorem is a remarkable phenomenon originated in the work of B. Kashin related to approximation theory [13], developed by S. Szarek into a general method [34] and carried over to all convex bodies by S. Szarek and N. Tomczak-Jaegermann ([40], see [28, §6]). The unit ball of L_p^n ($1 \leq p \leq \infty$) is denoted by B_p^n , i.e. for $p < \infty$

$$x \in B_p^n \text{ iff } |x(1)|^p + \dots + |x(n)|^p \leq n$$

and $x \in B_\infty^n$ if $\max_i |x(i)| \leq 1$. We choose to work with L_p^n spaces rather than ℓ_p^n spaces here because their unit balls have approximately unit volume: $c_1 \leq |B_p^n|^{1/n} \leq c_2$ for all $1 \leq p \leq \infty$.

Volume Ratio Theorem. (*Szarek, Tomczak-Jaegermann*). *Let K be a convex symmetric body in \mathbb{R}^n which contains B_2^n . Then for every integer $0 < k < n$*

there exists a subspace E of codimension k and such that

$$(4) \quad K \cap E \subseteq |CK|^{1/k} B_2^n.$$

In fact, the subspace E can be taken at random from the Grassmanian.

To obtain a coordinate version of the Volume Ratio Theorem, we can not just claim that (4) holds for some coordinate subspace $E = \mathbb{R}^I$: the octahedron $K = B_1^n$ forms an obstacle. However it turns out that the octahedron is the only obstacle, so our claim becomes true if one replaces the Euclidean ball B_2^n in (4) by its circumscribed octahedron B_1^n . This seems to be a general phenomenon when one passes from arbitrary to coordinate sections, see [30].

Theorem 1.3. *Let K be a convex symmetric body in \mathbb{R}^n which contains B_∞^n . Then for every integer $0 < k < n$ there exists a coordinate subspace E of codimension k and such that*

$$K \cap E \subseteq |CK|^{1/k} B_1^n.$$

This theorem follows from (3) by duality (Santaló and the reverse Santaló inequalities, the latter due to Bourgain and Milman).

Remarks. 1. The assumption $B_\infty^n \subset K$ of Theorem 1.3 is weaker than the assumption $B_2^n \subset K$ of the Volume Ratio Theorem. In fact, this assumption can be completely eliminated if one replaces $|CK|^{1/k}$ by the quantity

$$A_k(K) = \max \left(\frac{|CK|}{|K \cap E|} \right)^{1/\text{codim } E}$$

where the maximum is over the coordinate subspaces E , $\text{codim } E \geq k$. Clearly, $A_k(K) \leq |CK|^{1/k}$ if K contains B_∞^n . We will discuss this ‘‘Coordinate Volume Ratio Theorem’’ as well as the quantity $A_k(K)$ in more detail in Section 3.

2. The right dependence on k/n in the Volume Ratio Theorem and in Theorem 1.3 is a delicate problem. $|CK|^{1/k} = C^{n/k} |K|^{1/k}$, and while the factor $|K|^{1/k}$ is sharp (which is easily seen for ellipsoids or parallelepipeds), the exponential factor $C^{n/k}$ is *not*. We will improve it (in the dual form) to a linear factor Cn/k in Section 4.

Another example of applications of Theorem 1.1 is a coordinate version of Milman’s duality of diameters of sections. For a symmetric convex body K in \mathbb{R}^n , let

$$b_k(K) = \min \text{diam}(K \cap E_k),$$

where the minimum is over all k -dimensional subspaces E_k . Then for every $\varepsilon > 0$ and for any two positive integers k and m satisfying $k+m \leq (1-\varepsilon)n - C$ one has

$$(5) \quad b_k(K) b_m(K^\circ) \leq C/\varepsilon.$$

(in fact, this holds for random subspaces E_k in the Grassmanian) [22], [23]. This phenomenon reflects deep linear duality relations and provides a key tool in understanding the “global” duality in asymptotic convex geometry, see [23], [24].

To establish a version of this result for coordinate subspaces E_k , we have (as before) to change the metric that defines the diameter to that given by the octahedron circumscribed around the unit Euclidean ball (rather than the Euclidean ball itself). Then for the new diameter diam_1 we let

$$r_k(K) = \min \text{diam}_1(K \cap E_k),$$

where the minimum is over all k -dimensional *coordinate* subspaces E_k . In other words, the inequality $r_k(K) \leq 2r$ holds iff one can find a k -element set I so that one has $\sum_{i \in I} |x(i)| \leq r\sqrt{n}$ for all $x \in K$.

Theorem 1.4 (Duality for diameters of coordinate sections). *Let K be a symmetric convex body in \mathbb{R}^n . For any $\varepsilon > 0$ and for any two positive integers k and m satisfying $k + m \leq (1 - \varepsilon)n$ one has*

$$r_k(K) r_m(K^\circ) \leq C^{1/\varepsilon}.$$

In particular, there exists a subset of coordinates I of size, say, $\lceil n/3 \rceil$ such that the absolute values of the coordinates in I sum to at most $C\sqrt{n}$ either for all vectors in K or for all vectors in K° .

Remark. In most of the results of this paper, the convexity of K can be relaxed to a weaker *coordinate convexity*, see e.g. [17].

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2. SAUER-SHELAH LEMMA IN \mathbb{Z}^n

In 1971-72, Vapnik and Chervonenkis [41], Sauer [31] and Perles and Shelah [32] independently proved the following well known result, which has found applications in a variety of areas ranging from logics to probability to computer science.

Theorem 2.1 (Sauer-Shelah Lemma). *If $A \subset \{0, 1\}^n$ has cardinality $\#A > \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}$, then there exists a coordinate projection P of rank larger than d and such that*

$$(6) \quad PA = P\{0, 1\}^n.$$

A short proof of Sauer-Shelah Lemma can be found e.g. in [6, §17]; for numerous variants of the Lemma see the bibliography in [12] as well as [2], [35], [36].

To bring Sauer-Shelah Lemma to geometry, we will have to generalize it to sets $A \subset \mathbb{Z}^n$. The case when such A is bounded by a parallelepiped, i.e. $A \subset \prod_{i=1}^n \{0, \dots, N_i\}$, is well understood by now, see [15], [3], [33], [12]. In this section we will prove a generalization of Sauer-Shelah Lemma to $A \subset \mathbb{Z}^n$ independent of any boundedness assumptions.

We start with a simpler result. An *integer box* is a subset of \mathbb{Z}^n of the form $\{a_1, b_1\} \times \dots \times \{a_n, b_n\}$ with $a_i \neq b_i \forall i$. Similarly one defines integer boxes in \mathbb{Z}^I , where $I \subseteq \{1, \dots, n\}$.

Theorem 2.2. *If $A \subset \mathbb{Z}^n$, then*

$$(7) \quad \#A \leq 1 + \sum_P \#(\text{integer boxes in } PA),$$

where the sum is over all coordinate projections P in \mathbb{R}^n .

Remark. Let $A \subset \{0, 1\}^n$. Since the only lattice box that can be contained in PA is $P\{0, 1\}^n$, Theorem 2.2 implies that

$$(8) \quad \#A \leq 1 + \#(P \text{ for which } PA = P\{0, 1\}^n).$$

This estimate is due to A.Pajor [27, Theorem 1.4]. Note that this quantity is bounded by $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}$, where d is the maximal rank of P for which $PA = P\{0, 1\}^n$. This immediately implies Sauer-Shelah Lemma.

The result that we really need for geometric applications is Theorem 2.2 for *integer cells*, which are integer boxes whose all sides equal 1. Although the number of integer cells in a convex body can in principle be estimated through the number of integer boxes, the dependence will not be linear – a cube $[0, M]^n$ contains M^n integer cells and $(\frac{1}{2}M(M+1))^n$ integer boxes. To obtain Theorem 2.2 for integer cells, we will have to prove a more accurate extension of Sauer-Shelah Lemma to \mathbb{Z}^n .

The crucial in our discussion will be the notion of *coordinate convexity* (see e.g. [17]), which is weaker than that of convexity.

Definition 2.3. *Let K be a set in \mathbb{R}^n . The coordinate convex hull of K consists of the points $x \in \mathbb{R}^n$ such that for every choice of signs $\theta \in \{-1, 1\}^n$ one can find $y \in K$ such that*

$$\begin{aligned} y(i) &\geq x(i) && \text{if } \theta(i) = 1, \\ y(i) &\leq x(i) && \text{if } \theta(i) = -1. \end{aligned}$$

K is called *coordinate convex* if it coincides with its coordinate convex hull.

By changing \mathbb{R}^n to \mathbb{Z}^n the coordinate convexity can also be defined for subsets of \mathbb{Z}^n . Also, changing \mathbb{R}^n to \mathbb{R}^I and $\{-1, 1\}^n$ to $\{-1, 1\}^I$, the coordinate convexity is defined for subsets of \mathbb{R}^I (and similarly for \mathbb{Z}^I), where $I \subset \{1, \dots, n\}$.

One obtains a general convex body in \mathbb{R}^n by cutting off half-spaces. Similarly, a general coordinate convex body in \mathbb{R}^n is obtained by cutting off octants, i.e. translates of the sets $\theta \cdot \mathbb{R}_+^n$ with $\theta \in \{-1, 1\}^n$. Clearly, every convex set is coordinate convex; the converse is not true, as the cross shows $\{(x, y) \mid x = 0 \text{ or } y = 0\}$ in \mathbb{R}^2 .

The central combinatorial result of this section is the following theorem which we will prove after some comments.

Theorem 2.4. *For every $A \subset \mathbb{Z}^n$,*

$$(9) \quad \#A \leq 1 + \sum_P \#(\text{integer cells in } \text{cconv}PA),$$

where the sum is over all coordinate projections P .

The combinatorial dimension and Sauer-Shelah type results. Like Theorem 2.2, Theorem 2.4 also contains Sauer-Shelah Lemma: every subset $A \subset \{0, 1\}^n$ is coordinate convex, and the only lattice box that can be contained in PA is $P\{0, 1\}^n$, which implies (8) and hence Sauer-Shelah lemma.

To see a relation of Theorem 2.4 to later generalizations of Sauer-Shelah lemma, let us recall an important concept of the combinatorial dimension, which originates in the statistical learning theory and which became useful in convex geometry, combinatorics and analysis, see [1], [7], [39], [20], [30].

Definition 2.5. *The combinatorial dimension $v(A)$ of a set $A \subset \mathbb{R}^n$ is the maximal rank of a coordinate projection P such that $\text{cconv}(PA)$ contains some translate of the unit cube $P\{0, 1\}^n$.*

For $t > 0$, the scale-sensitive version of the combinatorial dimension is defined as $v(A, t) = v(t^{-1}A)$.

Equivalently, a subset $I \subset \{1, \dots, n\}$ is called t -shattered by A if there exists an $h \in \mathbb{R}^n$ such that, given any partition $I = I^- \cup I^+$, one can find an $x \in A$ such that $x(i) \leq h(i)$ if $i \in I^-$ and $x(i) \geq h(i) + t$ if $i \in I^+$. The combinatorial dimension $v(A, t)$ is the maximal cardinality of a subset t -shattered by A .

A few words on the history of the concept of the combinatorial dimension. For sets $A \subset \{0, 1\}^n$, the combinatorial dimension $v(A)$ is the classical *Vapnik-Chernovenkis dimension*; see [18] for a nice introduction to this important concept. For sets $A \subset \mathbb{Z}^n$, the notion of the combinatorial dimension $v(A)$ goes back to 1982-83, when Pajor used it for origin symmetric classes in view of applications to the local theory of Banach spaces [26]. He proved early versions of Sauer-Shelah Lemma for sets $A \subset \{0, \dots, p\}^n$ (see [26], [27, Lemma

4.9]). Pollard gave an explicit definition of $v(A)$ in his 1984 book on stochastic processes [29]. Haussler also discussed this concept in his 1989 work in learning theory ([11], see [12] and the references therein).

For convex and origin symmetric sets $A \subset \mathbb{R}^n$, the combinatorial dimension $v(A, t)$ is easily seen to coincide with the maximal rank of the coordinate projection PA of A that contains the centered coordinate cube of side t . In view of this straightforward connection to convex geometry and thus to the local theory of Banach spaces, the combinatorial dimension was a central quantity in the 1982-83 works of Pajor ([25], see Chapter IV of [27]). Connections of $v(A, t)$ to Gaussian processes and further applications to Banach space theory were established in the far reaching 1992 paper of M. Talagrand ([37], see also [39]). The quantity $v(A, t)$ was formally defined in 1994 by Kearns and Schapire for general sets A in their paper in learning theory [14].

Since its invention, the combinatorial dimension turned out to be very effective in measuring the complexity of a set A in combinatorics, functional analysis, statistical learning theory, the theory of empirical processes, discrete and convex geometry (see [1], [39], [20], [30]). Alternative names for the combinatorial dimension used in the literature on combinatorics and statistical learning theory are: Pollard dimension and pseudo dimension for $v(A)$, shattering and fat-shattering dimension for $v(A, t)$, see [12] and [19].

Similarly, *Natarajan dimension* $n(A)$ of a set $A \subset \mathbb{Z}^n$ is the maximal rank of a coordinate projection P such that PA contains an integer box (see [12]).

Theorems 2.2 and 2.4 easily imply two recent results of Haussler and Long [12] on the combinatorial and Natarajan dimensions, which are in turn generalizations of Sauer-Shelah lemma.

Corollary 2.6 (Haussler, Long [12]). *Let $A \subset \prod_{i=1}^n \{0, \dots, N_i\}$. Then*

(i) *We have*

$$|A| \leq \sum_{\#I \leq v(A)} \prod_{i \in I} N_i,$$

where the sum is over all subsets $I \subset \{1, \dots, n\}$ of cardinality at most $v(A)$ (we include $I = \emptyset$ and assign to it the summand equal to 1).

(ii) *In particular, if $A \subset \{0, \dots, N\}^n$ then*

$$|A| \leq \sum_{i=0}^{v(A)} \binom{n}{i} N^i.$$

(iii) *We have*

$$|A| \leq \sum_{\#I \leq n(A)} \prod_{i \in I} \binom{N_i + 1}{2}.$$

where the sum is over all subsets $I \subset \{1, \dots, n\}$ of cardinality at most $n(A)$ (we include $I = \emptyset$ and assign to it the summand equal to 1).

Proof. For (i), apply Theorem 2.4. All the summands in (9) that correspond to $\text{rank } P > v(A)$ vanish by the definition of the combinatorial dimension. Each of the non-vanishing summands is bounded by the number of integer cells in $\text{cconv } PA \subset P(\prod_{i=1}^n \{0, \dots, N_i\})$. This establishes (i) and thus (ii).

Repeating this for (iii), we only have to note that the number of integer boxes in $P(\prod_{i=1}^n \{0, \dots, N_i\}) = \{0\} \times \prod_{i \in I} \{0, \dots, N_i\}$ is at most $\binom{N_i+1}{2}$. ■

Remark. All the statements in Corollary 2.6 reduce to Sauer-Shelah lemma if $N_i = 1 \forall i$.

The proof. Here we prove Theorem 2.4. Define the *cell content* of A as

$$\Sigma(A) = \sum_P \#(\text{integer cells in } \text{cconv } PA),$$

where we include in the counting one 0-dimensional projection P (onto \mathbb{R}^\emptyset), for which the summand is set to be 1 if A is nonempty and 0 otherwise. This definition appears in [30]. We partition A into sets A_k , $k \in \mathbb{Z}$, defined as

$$A_k = \{x \in A : x(1) = k\}.$$

Lemma 2.7. For every $A \subset \mathbb{Z}^n$,

$$\Sigma(A) \geq \sum_{k \in \mathbb{Z}} \Sigma(A_k).$$

Proof. A cell \mathcal{C} in \mathbb{R}^I , $I \subset \{1, \dots, n\}$, will be considered as an ordered pair (\mathcal{C}, I) . This also applies to the trivial cell $(0, \emptyset)$ which we will include in the counting throughout this argument. The coordinate projection onto \mathbb{R}^I will be denoted by P_I .

We say that A has a cell (\mathcal{C}, I) if $\mathcal{C} \subset \text{cconv } P_I B$. The lemma states that A has at least as many cells as all the sets A_k have in total.

If A_k has a cell (\mathcal{C}, I) then A has it, too. Assume that $N > 1$ sets among A_k have a nontrivial cell (\mathcal{C}, I) . Since the first coordinate of any point in such a set A_k equals k , one necessarily has $1 \notin I$. Then $P_{\{1\} \cup I} A_k = \{k\} \times P_I A_k$, where the factor $\{k\}$ means of course the first coordinate. Hence

$$\begin{aligned} \{k\} \times \mathcal{C} &\subset \{k\} \times \text{cconv}(P_I A_k) = \text{cconv}(\{k\} \times P_I A_k) \\ &= \text{cconv } P_{\{1\} \cup I} A_k \subset \text{cconv } P_{\{1\} \cup I} A. \end{aligned}$$

Therefore the set $\text{cconv } P_{\{1\} \cup I} A$ contains the integer box $\{k_1, k_2\} \times \mathcal{C}$, where k_1 is the minimal k and k_2 is the maximal k for the N sets A_k . Then $\text{cconv } P_{\{1\} \cup I} A$ must also contain $\text{cconv}(\{k_1, k_2\} \times \mathcal{C}) \supset [k_1, k_2] \times \mathcal{C}$ which in turn contains at

least $k_2 - k_1 \geq N - 1$ integer cells of the form $\{a, a + 1\} \times \mathcal{C}$. Hence, in addition to one cell \mathcal{C} , the set A has at least $N - 1$ cells of the form

$$(10) \quad (\{a, a + 1\} \times \mathcal{C}, \{1\} \cup I).$$

Since the first coordinate of all points in any fixed A_k is the same, none of A_k may have a cell of the form (10). Note also that the argument above works also for the trivial cell.

This shows that there exists an injective mapping from the set of the cells that at least one A_k has into the set of the cells that A has. The lemma is proved. \blacksquare

Proof of Theorem 2.4. It is enough to show that for every $A \subset \mathbb{Z}^n$

$$\#A \leq \Sigma(A).$$

This is proved using Lemma 2.7 by induction on the dimension n .

The claim is trivially true for $n = 0$ (in fact also for $n = 1$). Assume it is true for some $n \geq 0$. Apply Lemma 2.7 and note that each A_k is a translate of a subset in \mathbb{Z}^{n-1} . We have

$$\Sigma(A) \geq \sum_{k \in \mathbb{Z}} \Sigma(A_k) \geq \sum_{k \in \mathbb{Z}} \#A_k = \#A$$

(here we used the induction hypothesis for each A_k). This completes the proof. \blacksquare

Volume and lattice cells. Now we head to Theorem 1.1.

Corollary 2.8. *Let K be set in \mathbb{R}^n . Then*

$$|\frac{1}{2}K| \leq 1 + \sum_P \#(\text{integer cells in } \text{cconv}PK),$$

where the sum is over all coordinate projections P .

For the proof we need a simple fact:

Lemma 2.9. *For every set K in \mathbb{R}^n and every $x \in \mathbb{R}^n$,*

$$\#(\text{integer cells in } x + K) \leq \#(\text{integer cells in } 2K).$$

Proof. The proof reduces to the observation that every translate of the cube $[0, 2]^n$ by a vector in \mathbb{R}^n contains an integer cell. This in turn is easily seen by reducing to the one-dimensional case. \blacksquare

Proof of Corollary 2.8. Let x be a random vector uniformly distributed in $[0, 1]^n$, and let $A_x = (x + K) \cap \mathbb{Z}^n$. Then $\mathbb{E}\#A_x = |K|$. By Theorem 2.4,

$$(11) \quad |K| \leq 1 + \mathbb{E} \sum_P \#(\text{integer cells in } \text{cconv}PA_x),$$

while

$$(12) \quad \text{cconv}PA_x \subset \text{cconv}P(x + K) = Px + \text{cconv}PK.$$

By this and Lemma 2.9,

$$\#(\text{integer cells in } \text{cconv}PA_x) \leq \#(\text{integer cells in } \text{cconv}P(2K)).$$

Thus by (11)

$$|K| \leq 1 + \sum_P \#(\text{integer cells in } \text{cconv}P(2K)).$$

This proves the corollary. ■

Remark. The proof of Theorem 2.2 is very similar and in fact is simpler than the argument above. One looks at $\Sigma(A) = \sum_P \#(\text{integer boxes in } PA)$ and repeats the proof without worrying about coordinate convexity.

Now we can prove the main geometric result of this section.

Theorem 2.10. *Let K be a set in \mathbb{R}^n . Then there exists a coordinate projection P in \mathbb{R}^n such that $\text{cconv}PK$ contains at least $|\frac{1}{4}K| - 2^{-n}$ integer cells.*

Proof. By Corollary 2.8,

$$|\frac{1}{2}K| \leq 1 + (2^n - 1) \max_P \#(\text{integer cells in } \text{cconv}PK).$$

Hence $\max_P \#(\text{integer cells in } \text{cconv}PK) \geq |\frac{1}{4}K| - 2^{-n}$. ■

Note that $|\frac{1}{4}K| - 2^{-n} \geq |\frac{1}{6}K|$ if $|\frac{1}{6}K| \geq 1$. This implies Theorem 1.1.

3. THE COORDINATE VOLUME RATIO THEOREM

Let K be a set in \mathbb{R}^k . For $0 < k < n$, define

$$A_k(K) = \max \left(\frac{|CK|}{|K \cap E|} \right)^{1/\text{codim } E}$$

where the maximum is over the coordinate subspaces E , $\text{codim } E \geq k$, and $C > 0$ is an absolute constant whose value will be discussed later.

Theorem 3.1 (Coordinate Volume Ratio Theorem). *Let K be a convex symmetric set in R^n . Then for every integer $0 < k < n$ there exists a coordinate section E , $\text{codim } E = k$, such that*

$$K \cap E \subset A_k(K) B_1^n.$$

The proof relies on the extension on Sauer-Shelah Lemma in \mathbb{Z}^n from the previous section and on the duality for the volume, which is Santalo and the reverse Santalo inequalities (the latter due to J.Bourgain and V.Milman). We will prove Theorem 3.1 in the end of this section.

1. In the important case when K contains the unit cube, we have $A_k(K) \leq |CK|^{1/k}$. This implies:

Corollary 3.2. *Let K be a convex body in \mathbb{R}^n which contains the unit cube B_∞^n . Then for every integer $0 < k < n$ there exists a coordinate subspace E of codimension k and such that*

$$K \cap E \subseteq |CK|^{1/k} B_1^n.$$

The assumptions of this corollary are weaker than those of the classical Volume Ratio Theorem stated in the introduction, because the cube B_∞^n is inscribed into the Euclidean ball B_2^n . The conclusion of Corollary 3.2 is that some coordinate section $K \cap E$ is bounded by the octahedron B_1^n , which is circumscribed around the Euclidean ball B_2^n . No stronger conclusion is for a *coordinate* section is possible: $K = B_1^n$ itself is an obstacle.

Nevertheless, by a result of Kashin ([13], see a sharper estimate in Garnaev-Gluskina [8]) a random section of B_1^n in the Grassmanian $G_{n,k}$ with $k = \lceil n/2 \rceil$ is equivalent to the Euclidean ball B_2^k . Thus a random (no longer coordinate) section of $K \cap E$ of dimension, say, $\frac{1}{2} \dim(K \cap E)$ will already be a subset of $|CK|^{1/k} B_2^n$. This shows that Corollary 3.2 is close in nature to the classical Volume Ratio Theorem. It gives *coordinate* subspaces without sacrificing too much of the power of the Volume Ratio Theorem.¹

In the next section we will prove a (dual) result even sharper than Corollary 3.2.

2. The quantity $A_k(K)$ is best illustrated on the example of classical bodies. If K is the parallelpiped $\prod_{i=1}^n [-a_i, a_i]$ with semiaxes $a_1 \geq a_2 \geq \dots \geq a_n > 0$, then

$$(13) \quad A_k(K) = (2C)^{n/k} \left(\prod_{i=1}^k a_i \right)^{1/k},$$

a quantity proportional to the geometric mean of the largest k semiaxes. The same holds if K is the ellipsoid with the coordinate nonincreasing semiaxes $a_i \sqrt{n}$, i.e. $x \in K$ iff $\sum_{i=1}^n x(i)^2 / a_i^2 \leq n$. This is clearly better than

$$|CK|^{1/k} = (2C)^{n/k} \left(\prod_{i=1}^n a_i \right)^{1/k},$$

which appears in the classical Volume Ratio Theorem (note that the inclusion $B_2^n \subset K$ implies in the ellipsoidal example that all $a_i \geq 1$.)

¹Even though in the Coordinate Volume Ratio Theorem the coordinate section can not be random in general, a very recent work of Giannopoulos, Milman and Tsolomitis [10] and of the author [42] suggests that one can automatically regain randomness of a bounded section in the Grassmanian if one only knows the *existence* of a bounded section in the Grassmanian.

3. An important observation is that (13) holds for *arbitrary* symmetric convex body K , in which case $a_i\sqrt{n}$ denote the semiaxes of an M-ellipsoid of K . The M-ellipsoid is a deep concept in the modern convex geometry; it nicely reflects volumetric properties of convex bodies. For every symmetric convex body K in \mathbb{R}^n there exists an ellipsoid \mathcal{E} such that $|K| = |\mathcal{E}|$ and K can be covered by at most $\exp(C_0n)$ translates of \mathcal{E} . Such ellipsoid \mathcal{E} is called an M-ellipsoid of K (with parameter C_0). Its existence (with the parameter equal to an absolute constant) was proved by V.Milman [21]; for numerous consequences see [28], [24], [9].

Fact 3.3. *Let K be a symmetric convex body in \mathbb{R}^n and \mathcal{E} be its M-ellipsoid with parameter C_0 . Then*

$$A_k(K) \leq (CC_0)^{n/k} \left(\prod_{i=1}^k a_i \right)^{1/k},$$

where $a_i\sqrt{n}$ are the semiaxes of \mathcal{E} in a nondecreasing order. In other words, a_i are the singular values of a linear operator that maps B_2^n onto \mathcal{E} .

Proof. The fact that \mathcal{E} is an M-ellipsoid of K implies by standard covering arguments that $(CC_0)^n |K \cap E| \geq |\mathcal{E} \cap E|$ for all subspaces E in \mathbb{R}^n , see e.g. [16, Fact 1.1(ii)]. Since $|K| = |\mathcal{E}|$, we have $A_k(K) \leq (CC_0)^{n/k} A_k(\mathcal{E})$, which reduces the problem to the examples of ellipsoids discussed above. ■

4. A quantity similar to $A_k(K)$ and which equals $(\prod_{i=1}^{l+k} a_i)^{1/k}$ for the ellipsoid with nonincreasing semiaxes a_i plays a central role in the recent work of Mankiewicz and Tomczak-Jaegermann [16]. They proved a volume ratio-type result for this quantity (for random non-coordinate subspaces E in the Grassmanian) which works for $\dim E \leq n/2$.

5. Theorem 3.1 follows from its more general dual counterpart that allows to compute the combinatorial dimension of a set in terms of its volume.

Let K be a set in \mathbb{R}^n . For $0 < k < n$, define

$$a_k(K) = \min \left(\frac{|cK|}{|P_E K|} \right)^{1/\text{codim } E}$$

where the minimum is over the coordinate subspaces E , $\text{codim } E \geq k$, and $c > 0$ is an absolute constant whose value will be discussed later.

Theorem 3.4. *Let K be a convex set in \mathbb{R}^n . Then for every integer $0 < k < n$,*

$$v(K, a_k(K)) \geq n - k.$$

Proof. By applying an arbitrarily small perturbation to K , we can assume that the function $R \mapsto v(RK, 1)$ maps \mathbb{R}_+ onto $\{0, 1, \dots, n\}$. Let R be a solution to the equation

$$v(RK, 1) = n - k.$$

By Corollary 2.8,

$$(14) \quad \left| \frac{1}{2}RK \right| \leq 1 + \max_P \#(\text{integer cells in } P(RK))$$

where the maximum is over all coordinate projections P in \mathbb{R}^n . Since $v(RK, 1) \geq 1$, the maximum in (14) is at least 1. Hence there exists a coordinate projection $P = P_E$ onto a coordinate subspace E such that

$$\left| \frac{1}{2}RK \right| \leq 2\#(\text{integer cells in } P_E(RK)).$$

Since the number of integer cells in a set is bounded by its volume,

$$R^n \left| \frac{1}{2}K \right| \leq 2|P_E(RK)| \leq 2R^{n-l}|P_EK|$$

where $n - l = \dim E$. It follows that

$$\frac{1}{R} \geq \left(\frac{|\frac{1}{4}K|}{|P_EK|} \right)^{1/l} \quad \text{and} \quad v(K, \frac{1}{R}) = n - k.$$

It only remains to note that by the maximal property of the combinatorial dimension, $n - l = \dim E \leq n - k$; thus $l = \text{codim } E \geq k$. \blacksquare

Lemma 3.5. *For every integer $0 < k < n$, we have $A_k(K) a_k(nK^\circ) \geq 1$.*

Proof. Let $L = nK^\circ$. Fix numbers $0 < k \leq l < n$ and a coordinate subspace E , $\text{codim } E = l$. Santalo and the reverse Santalo inequalities (the latter due to Bourgain and Milman [5], see [28] §7) imply that

$$\begin{aligned} |L| &\geq c_1^n |K|^{-1}, \\ |P_E L| &\leq \left(\frac{C_1}{n-l} \right)^{n-l} |L^\circ \cap E|^{-1} = \left(\frac{C_1 n}{n-l} \right)^{n-l} |K \cap E|^{-1}. \end{aligned}$$

Then

$$\left(\frac{|cL|}{|P_E L|} \right)^{1/l} \geq \left[(c_1 c)^n \left(\frac{n-l}{C_1 n} \right)^{n-l} \frac{|K \cap E|}{|K|} \right]^{1/l} \geq \left(\frac{|K \cap E|}{|(C_2/c)K|} \right)^{1/l}.$$

Now take the minimum over $l \geq k$ and over E to see that $a_k(L) \geq A_k(K)^{-1}$ if we choose $C = C_2/c$. \blacksquare

Remark. Theorem 3.4 holds for general sets K (not necessarily convex) if in the definition of $a_k(K)$ one replaces $|P_E K|$ by $|\text{cconv } P_E K|$. The proof above easily modifies.

Proof of Theorem 3.1. By Theorem 3.4 and Lemma 3.5,

$$v(K^\circ, (nA_k(K))^{-1}) = v(nK^\circ, A_k(K)^{-1}) \geq n - k.$$

By the symmetry of K , this means that exists an orthogonal projection P_E onto a coordinate subspace E , $\text{codim } E = k$, such that

$$P_E(K^\circ) \supset P_E((nA_k(K))^{-1}[-\frac{1}{2}, \frac{1}{2}]^n).$$

Dualizing, we obtain

$$K \cap E \subset 2A_k(K)B_1^n.$$

The constant 2 can be removed by increasing the value of the absolute constant C in the definition of $A_k(K)$. ■

4. VOLUMES OF THE SETS IN THE L_p BALLS

The classical Volume Ratio Theorem stated in the introduction is sharp up to an absolute constant C (see e.g. [28] §6). However, if we look at the factor $|CK|^{1/k} = C^{n/k}|K|^{1/k}$ which also appears in Corollary 1.3, then it becomes questionable whether the exponential dependence of the proportion n/k is the right one. We will improve it in the dual setting to a *linear* dependence. The main result of this section computes the combinatorial dimension of a set K (not even convex) in \mathbb{R}^n in terms of its volume restricted to B_p^n . In other words, we are looking at the probability measure defined as

$$\mu_p(K) = \frac{|K \cap B_p^n|}{|B_p^n|}.$$

Theorem 4.1. *Let K be a set in \mathbb{R}^n and $1 \leq p \leq \infty$. Then for every integer $0 < k \leq n$ one has*

$$(15) \quad v(K, t) \geq n - k \quad \text{for} \quad t = c \binom{k}{n} \mu_p(K)^{1/k}.$$

Remarks. 1. The result is sharp up to an absolute constant c . An appropriate example will be given after the proof.

2. Corollary 3.2 is an immediate consequence of Theorem 4.1 by duality.

3. To compare Theorem 4.1 to the classical Volume Ratio Theorem, one can read (15) for convex bodies as follows:

- (*) There exists a coordinate projection P of rank $n - k$ so that PK contains a translate of the cube $P(tB_\infty^n)$ with $t = c \binom{k}{n} \mu_p(K)^{1/k}$,

while the classical Volume Ratio Theorem states that

- (**) There is a random orthogonal projection P of rank $n - k$ so that PK contains a translate of the ball $P(tB_2^n)$ with $t = c^{n/k} \mu_2(K)^{1/k}$.

Beside the central fact of the existence of a *coordinate* projection in (*), note also the linear dependence on the proportion k/n (in contrast to the exponential dependence in (**)), and also the arbitrary p .

For the proof of Theorem 4.1, we will need to know that the volumes $w_p(n) = |B_p^n|$ approximately increase in n .

Lemma 4.2. $w_p(k) \leq Cw_p(n)$ provided $k \leq n$.

Proof. We have

$$w_p(k) = k^{k/p} \frac{(2\Gamma(1 + \frac{1}{p}))^k}{\Gamma(1 + \frac{k}{p})},$$

see [28] (1.17). Note that

$$a^{1/p} := 2\Gamma(1 + \frac{1}{p}) \geq 2 \min_{x>0} \Gamma(x) \geq 1.76.$$

We then use Stirling's formula

$$\Gamma(1+z) \approx e^{-z} z^{z+1/2}$$

where $a \approx b$ means $ca \leq b \leq Cb$ for some absolute constants $c, C > 0$.

Consider two cases.

1. $k \geq p$. We have

$$(16) \quad w_p(k) = \frac{(ak)^{k/p}}{\Gamma(1 + \frac{k}{p})} \approx (ak)^{k/p} e^{k/p} \left(\frac{k}{p}\right)^{-\frac{k}{p} - \frac{1}{2}} \approx (eap)^{k/p} \sqrt{\frac{p}{k}}.$$

2. $k \leq p$. In this case $\Gamma(1 + \frac{k}{p}) \approx 1$, thus

$$(17) \quad w_p(k) \approx (ak)^{k/p}.$$

To complete the proof, we consider three possible cases.

(a) $k \leq n \leq p$. Here the lemma is trivially true by (17).

(b) $k \leq p \leq n$. Here

$$\begin{aligned} \frac{w_p(n)}{w_p(k)} &\gtrsim \frac{(eap)^{n/p}}{(ak)^{k/p}} \sqrt{\frac{p}{n}} \geq a^{\frac{n-k}{p}} \sqrt{\frac{k}{n}} \quad (\text{because } p \geq k) \\ &\geq (1.76)^{n-k} \sqrt{\frac{k}{n}} \geq c > 0. \end{aligned}$$

(c) $p \leq k \leq n$. Here

$$\frac{w_p(n)}{w_p(k)} \gtrsim (eap)^{\frac{n-k}{p}} \sqrt{\frac{n}{k}}.$$

Since $ep > 1$, one finishes the proof as in case (b). ■

Proof of Theorem 4.1. We can assume that $K \subseteq B_p^n$. Let

$$u^n = \frac{|K|}{|B_p^n|}.$$

By applying an arbitrarily small perturbation of K we can assume that the function $R \mapsto v(RK, 1)$ maps \mathbb{R}_+ onto $\{0, 1, \dots, n\}$. Then there exists a solution R to the equation

$$v(RK, 1) = n - k.$$

The geometric results of the previous sections, such as Corollary 2.8 and Theorem 2.10, contain absolute constant factors which would destroy the linear dependence on k/n . So we have to be more careful and apply (11) together with (12) instead:

$$(18) \quad |RK| \leq 1 + \max_{x \in (0,1)^n} \sum_P \#(\text{integer cells in } Px + \text{cconv}P(RK)).$$

Since $v(RK, 1) > 0$, there exists a coordinate projection P such that

$$\max_{x \in (0,1)^n} \max_P \#(\text{integer cells in } Px + \text{cconv}P(RK)) \geq 1.$$

Hence the maximum in (18) is bounded below by 1 (for $x = 0$). Thus

$$\begin{aligned} |RK| &\leq 2 \max_{x \in (0,1)^n} \sum_P \#(\text{integer cells in } Px + \text{cconv}P(RK)) \\ &\leq 2 \max_{x \in (0,1)^n} \sum_{d=1}^{n-k} \sum_{\text{rank } P=d} \#(\text{integer cells in } Px + \text{cconv}P(RK)) \\ &\leq 2 \sum_{d=1}^{n-k} \sum_{\text{rank } P=d} |\text{cconv}P(RK)| \end{aligned}$$

because the number of integer cells in a set is bounded by its volume. Note that $\text{cconv}P(RK) \subset \text{conv}P(RK) \subset RP(B_p^n)$ by the assumption. Then denoting by P_d the orthogonal projection in \mathbb{R}^n onto \mathbb{R}^d , we have

$$|RK| \leq 2 \sum_{d=1}^{n-k} \binom{n}{d} R^d |P_d B_p^n|.$$

Now note that $P_d B_p^n = (n/d)^{1/p} B_p^d$. Hence

$$(19) \quad |RK| \leq 2 \sum_{d=1}^{n-k} \binom{n}{d} \left(\frac{n}{d}\right)^{d/p} R^d w_p(d).$$

Now $|RK| = R^n|K| = R^n u^n w_p(n)$ in the left hand side of (19) and $w_p(d) \leq C w_p(n)$ in the right hand side of (19) by Lemma 4.2. After dividing (19) through by $R^n w_p(n)$ we get

$$(20) \quad u^n \leq 2C \sum_{d=1}^{n-k} \binom{n}{d} \left(\frac{n}{d}\right)^{d/p} R^{d-n}.$$

Let $0 < \varepsilon < 1$. There exists a $1 \leq d \leq n - k$ such that

$$\left(\frac{n}{d}\right)^{d/p} R^{d-n} \geq (2C)^{-1} \varepsilon^{n-d} (1 - \varepsilon)^d u^n;$$

otherwise (20) would fail by the Binomial Theorem. From this we get

$$R \leq (2C)^{\frac{1}{n-d}} \left(\frac{n}{d}\right)^{\frac{d}{p(n-d)}} \frac{1}{\varepsilon} \left(\frac{1}{1 - \varepsilon}\right)^{\frac{d}{n-d}} u^{-\frac{n}{n-d}}.$$

Define δ by the equation $d = (1 - \delta)n$. We have

$$R \leq (2C)^{1/\delta n} [(1 - \delta)^{1/p} (1 - \varepsilon)]^{-\frac{1-\delta}{\delta}} \frac{1}{\varepsilon} u^{-1/\delta}.$$

Now we use this with ε defined by the equation $n - k = (1 - \varepsilon)n$. Since $d \leq n - k$, we have $\varepsilon \leq \delta$, so

$$[(1 - \delta)^{1/p} (1 - \varepsilon)]^{-\frac{1-\delta}{\delta}} \leq (1 - \varepsilon)^{\frac{2(1-\varepsilon)}{\varepsilon}} < C \quad \text{for } 0 < \varepsilon < 1.$$

Thus

$$R \leq \frac{C}{\varepsilon} u^{-1/\varepsilon}.$$

Then for $t := C^{-1} \varepsilon u^{1/\varepsilon} \leq \frac{1}{R}$ we have $v(K, t) \geq v(K, \frac{1}{R}) = n - k$. ■

Example. For every integer $n/2 \leq k < n$ there exists a coordinate convex body K in \mathbb{R}^n of arbitrarily small volume and such that for all $1 \leq p \leq n$

$$v(K, t) > n - k \quad \text{implies} \quad t < C \left(\frac{k}{n}\right) \mu_p(K)^{1/k}.$$

Proof. Fix an $\varepsilon > 0$ and let K be the set of all points $x \in B_p^n$ such that one has $|x(i)| \leq \varepsilon$ for at least k coordinates $i \in \{1, \dots, n\}$. Then K contains $\binom{n}{k}$ disjoint sets K_A indexed by $A \subset \{1, \dots, n\}$, $|A| = k$,

$$K_A = \{x \in B_p^n : \text{one has } |x(i)| \leq \varepsilon \text{ iff } i \in A\}.$$

For each A , write

$$K_A = ([-\varepsilon, \varepsilon]^A \times (\varepsilon I)^{A^c}) \cap B_p^n$$

where $I = (-\infty, -1) \cup (1, \infty)$. In the next line we use notation $f(\varepsilon) \asymp g(\varepsilon)$ if $f(\varepsilon)/g(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$ uniformly over $p \in [1, \infty]$. We have

$$\begin{aligned} |K_A| &\asymp |([- \varepsilon, \varepsilon]^A \times \mathbb{R}^{A^c}) \cap B_p^n| \asymp |[- \varepsilon, \varepsilon]^A| \times |B_p^n \cap \mathbb{R}^{A^c}| \\ &= (2\varepsilon)^k \left| \left(\frac{n}{n-k} \right)^{1/p} B_p^{n-k} \right| \geq (2\varepsilon)^k |B_p^{n-k}|. \end{aligned}$$

Thus there exists an $\varepsilon = \varepsilon(n, k) > 0$ so that

$$\mu_p(K) = \binom{n}{k} \mu_p(K_A) \geq \binom{n}{k} (c\varepsilon)^k \frac{|B_p^{n-k}|}{|B_p^n|}.$$

Now we need now to bound below the ratio of the volumes.

CLAIM. $\frac{w_p(n-k)}{w_p(n)} \geq c^k$.

Consider two possible cases:

(a) $p \geq n - k$. In this case $n/2 \leq n - k \leq p \leq n$, and by (16) and (17) we have

$$\begin{aligned} \frac{w_p(n-k)}{w_p(n)} &= \frac{(a(n-k))^{\frac{n-k}{p}}}{(eap)^{\frac{n}{p}}} \sqrt{\frac{n}{p}} \\ &\geq \left(\frac{n-k}{ean} \right)^{\frac{n}{p}} \quad (\text{since } p \leq n) \\ &\geq \left(\frac{1}{2ea} \right)^2 \quad (\text{since } p \geq n-k \geq n/2) \end{aligned}$$

which proves the claim in this case.

(b) $p \leq n - k$. Here

$$\begin{aligned} \frac{w_p(n-k)}{w_p(n)} &= \frac{(eap)^{\frac{n-k}{p}}}{(eap)^{\frac{n}{p}}} \sqrt{\frac{n}{n-k}} \\ &\geq \frac{1}{2} (eap)^{-\frac{k}{p}} \quad (\text{since } n/2 \leq k \leq n) \\ &\geq c^k. \end{aligned}$$

This proves the claim.

We have thus shown that $\mu_p(K) \geq \binom{n}{k} (c\varepsilon)^k$, so

$$\mu_p(K)^{1/k} > c \left(\frac{n}{k} \right) \varepsilon.$$

On the other hand, no coordinate projection PK of dimension exceeding $n - k$ can contain a translate of the cube $P[-t, t]^n$ for $t > \varepsilon$. Thus

$$v(K, t) > n - k \quad \text{implies} \quad t \leq \varepsilon < C \left(\frac{k}{n} \right) \mu_p(K)^{1/k}.$$

Note also that the volume of K can be made arbitrarily small by decreasing ε . ■

The same example also works for $p = \infty$.

5. DUALITY FOR DIAMETERS OF COORDINATE SECTIONS

Here we prove Theorem 1.4. Formally,

$$r_k(K) = \frac{2}{\sqrt{n}} \min_{|I|=k} \max_{x \in K} \sum_{i \in I} |x(i)|.$$

Theorem 5.1. *Let K be a symmetric convex body in \mathbb{R}^n . For any $\varepsilon > 0$ and for any two positive integers k and m satisfying $k + m \leq (1 - \varepsilon)n$ one has*

$$r_k(K) r_m(K^\circ) \leq C^{1/\varepsilon}.$$

The proof is based on Corollary 3.2.

Proof. Define δ and λ as follows: $k = (1 - \delta)n$, $m = (1 - \lambda)n$. Then $\delta + \lambda - 1 > \varepsilon$. Let $t_1, t_2 > 0$ be parameters, and define

$$K_1 = \text{conv} \left(K \cup t_1 n^{-1/2} B_\infty^n \right) \cap \frac{1}{t_2} n^{-1/2} B_1^n.$$

Consider two possible cases:

(1) $|K_1| \leq |n^{-1/2} B_\infty^n|$. Since K_1 contains $t_1 n^{-1/2} B_\infty^n$, we have

$$\frac{\sqrt{n}}{t_1} K_1 \supset B_\infty^n \quad \text{and} \quad \left| \frac{\sqrt{n}}{t_1} K_1 \right| \leq \left| \frac{1}{t_1} B_\infty^n \right| = \left(\frac{2}{t_1} \right)^n.$$

Corollary 3.2 implies the existence of a subspace E , $\dim E = (1 - \delta)n$, such that

$$\frac{\sqrt{n}}{t_1} K_1 \cap E \subset \left(\frac{C}{t_1} \right)^{1/\delta} B_1^n.$$

Multiplying through by t_1/\sqrt{n} and recalling the definition of K_1 , we conclude that

$$(21) \quad K \cap E \cap \frac{1}{t_2} n^{-1/2} B_1^n \subset t_1 \left(\frac{C}{t_1} \right)^{1/\delta} n^{-1/2} B_1^n.$$

(2) $|K_1| > |n^{-1/2} B_\infty^n|$. Note that

$$K_1^\circ = \text{conv} \left[\left(K^\circ \cap \frac{1}{t_1} n^{-1/2} B_1^n \right) \cup t_2 n^{-1/2} B_\infty^n \right].$$

By Santaló and reverse Santaló inequalities,

$$|K_1^\circ| < |C n^{-1/2} B_1^n|.$$

Since K_1° contains $t_2 n^{-1/2} B_\infty^n$, we have

$$\frac{\sqrt{n}}{t_2} K_1^\circ \supset B_\infty^n \quad \text{and} \quad \left| \frac{\sqrt{n}}{t_2} K_1^\circ \right| \leq \left| \frac{C}{t_2} B_1^n \right| \leq \left(\frac{C_2}{t_2} \right)^n.$$

Arguing similarly to case (1) for K° , we find a subspace F , $\dim F = (1 - \lambda)n$, and such that

$$(22) \quad K^\circ \cap F \cap \frac{1}{t_1} n^{-1/2} B_1^n \subseteq t_2 \left(\frac{C}{t_2}\right)^{1/\lambda} n^{-1/2} B_1^n.$$

Looking at (21) and (22), we see that our choice of t_1, t_2 should be so that

$$t_1 \left(\frac{C}{t_1}\right)^{1/\delta} = \frac{1}{2t_2}, \quad t_2 \left(\frac{C}{t_2}\right)^{1/\lambda} = \frac{1}{2t_1}.$$

Solving this for t_1 and t_2 we get

$$\frac{1}{2t_1} = \frac{1}{\sqrt{2}} C^{\frac{\delta-\lambda+1}{\delta+\lambda-1}} =: R_1, \quad \frac{1}{2t_2} = \frac{1}{\sqrt{2}} C^{\frac{\lambda-\delta+1}{\delta+\lambda-1}} =: R_2.$$

Then (21) becomes

$$K \cap E \subseteq R_2 n^{-1/2} B_1^n$$

and (22) becomes

$$K^\circ \cap F \subseteq R_1 n^{-1/2} B_1^n.$$

It remains to note that

$$R_1 R_2 = \frac{1}{2} C^{2/(\delta+\lambda-1)} < \frac{1}{2} C^{2/\varepsilon}.$$

This completes the proof. ■

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, 1 SHIELDS AVE,
DAVIS, CA 95616, U.S.A.

E-mail address: `vershynin@math.ucdavis.edu`