KADISON-SINGER MEETS BOURGAIN-TZAFRIRI

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Abstract. We show that the Kadison-Singer problem is equivalent to the (strong) restricted invertibility conjecture of Bourgain-Tzafriri. We also show that these two problems are equivalent to two problems in frame theory holding simultaneously: (1) the Feichtinger Conjecture, and (2) the R_ϵ-Conjecture. Next, we show that Bourgain-Tzafriri restricted-invertibility principle holds on *random* subspaces. This extends the principle in several ways and shows that slightly weaker versions of the conjectures hold.

1. Introduction

If $\mathcal{A}$ is a subalgebra of the bounded linear operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ then a **em** is a linear functional $f \in \mathcal{A}^*$ for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (i.e. whenever T is a positive operator). The set of states is a convex subset of the dual space which is compact in the ω^*-topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the *pure states*. In 1959 Kadison and Singer [K-S] posed a problem now famous as the Kadison-Singer problem.

Kadison-Singer Problem. *Does every pure state on an atomic maximal abelian self-adjoint subalgebra of $B(\mathcal{H})$ extend uniquely to a pure state on $B(\mathcal{H})$?*

The space $B(\mathcal{H})$ is the most fundamental C^*-algebra and the Kadison-Singer Problem is an equally fundamental question concerning certain subalgebras of $B(\mathcal{H})$. Thus, this problem has generated a large body of literature over the years (see [W] and its references).

It has been known since [K-S] that the Kadison-Singer Problem can be reformulated in terms of complex matrices where it is known as the *paving conjecture*. We will denote $\mathbb{C}^N$ by $\mathcal{H}_N$ and let $\{e_i\}_{i=1}^N$ be the unit vector basis of $\mathcal{H}_N$. Given a subset I of the integers, we denote by P_I the orthogonal projection in ℓ_2 onto the subspace spanned by e_i, $i \in I$.

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Conjecture 1.1 (The Paving Conjecture). *For $\epsilon > 0$, there is a constant $M = M(\epsilon)$ such that for every integer n and every linear operator S on ℓ_2^n whose matrix has zero diagonal, one can find a partition $\{\sigma_j\}_{j=1}^M$ of $\{1, \cdots, n\}$, such that*

$$\|P_{\sigma_j} S P_{\sigma_j}\| \leq \varepsilon \|S\| \quad \textit{for all } j = 1, 2, \cdots, M.$$

A deep analysis of the paving conjecture was done by Bourgain and Tzafriri [B-Tz].

We are going to show that these two problems are equivalent to a strong form of the restricted invertibility problem of Bourgain-Tzafriri. This problem arose from a fundamental result of Bourgain-Tzafriri [BT] in 1987. Bourgain and Tzafriri [BT] proved the following result known as the *Restricted-Invertibility Theorem*:

Theorem 1.2 (Bourgain-Tzafriri). *There is a universal constant $c > 0$ so that whenever $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for $1 \leq i \leq n$, there exists a subset $\sigma \subset \{1, 2, \cdots, n\}$ of cardinality $|\sigma| \geq cn/\|T\|^2$ so that*

$$\Big\| \sum_{j \in \sigma} a_j T e_j \Big\|^2 \geq c \sum_{j \in \sigma} |a_j|^2,$$

for all choices of scalars $\{a_j\}_{j \in \sigma}$.

Theorem 1.2 gave rise to the following conjecture.

Conjecture 1.3 (Strong B-T). *There is a universal constant $c > 0$ so that for every $B > 0$ there is a natural number $M = M(B)$ so that if $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for all $1 \leq i \leq n$ and $\|T\| \leq B$, then there is a partition $\{I_j\}_{j=1}^M$ of $\{1, 2, \cdots, n\}$ so that for each $1 \leq j \leq M$ and all choices of scalars $\{a_i\}_{i \in I_j}$ we have:*

$$\Big\| \sum_{i \in I_j} a_i T e_i \Big\|^2 \geq c \sum_{i \in I_j} |a_i|^2.$$

In the next section we will prove

Theorem 1.4. *The Kadison-Singer Problem is equivalent to Strong B-T.*

Also, in section 3 we will show that the Kadison-Singer Problem is equivalent to both the Feichtinger Conjecture and the R_ϵ-Conjecture in Hilbert space frame theory holding at the same time.

In view of the celebrated power of the Probabilistic Method in geometric functional analysis, one naturally asks whether the Bourgain-Tzafriri Restricted-Invertibility Theorem 1.2 holds for a *random* subset σ. More generally, for a bounded linear operator T on ℓ_2 one looks at all *sets of the isomorphism* σ of

T, i.e. those subsets of the integers for which the equivalence[1]

$$\frac{1}{4}\sum_{i\in\sigma}\|a_iTe_i\|^2 \le \Big\|\sum_{i\in\sigma}a_iTe_i\Big\|^2 \le 4\sum_{i\in\sigma}\|a_iTe_i\|^2 \tag{1.1}$$

holds for all choices of scalars $\{a_j\}_{j\in\sigma}$. Then one asks, how large is the family $\Sigma(T)$ of all the sets of the isomorphism of T? The following dimension-free result will be proved in Section 4. It gives an asymptotically sharp bound on the average of the characteristic functions of the sets $\sigma\in\Sigma(T)$. This implies and extends in several ways the Bourgain-Tzafriri's Restricted-Invertibility Theorem 1.2.

Theorem 1.5. *Let T be a norm one linear operator on ℓ_2. Then there exists a probability measure ν on $\Sigma(T)$ such that*

$$\nu\{\sigma|\, i\in\sigma\} \ge c\|Te_i\|^2 \quad \textit{for all } i.$$

This result implies somewhat weaker versions of the Paving Conjecture and Strong B-T, as we will see in Section 4.

2. Kadison-Singer is equivalent to Strong B-T

We start by reformulating Strong B-T in terms of positive self-adjoint operators. We will only prove one implication of this which we need for our work but it can be shown that these are actually equivalent.

Proposition 2.1. *Strong B-T has a positive solution if the following holds. There is a universal constant $c>0$ so that for every positive self-adjoint $n\times n$-matrix S with ones on the diagonal there is a partition $\{\sigma_j\}_{j=1}^M$ of $\{1,2,\cdots,n\}$ with $M=M(\|S\|)$ so that for every $j=1,2,\cdots,M$ we have*

$$\langle SP_{\sigma_j}f, P_{\sigma_j f}\rangle = \|S^{1/2}P_{\sigma_j}f\|^2 \ge c\|P_{\sigma_j}f\|^2,$$

for all $f\in\mathcal{H}_n$.

Proof. Let $T:\ell_2^n\to\ell_2^n$ satisfy $\|Te_i\|=1$ for all $i=1,2,\cdots,n$. Set $Te_i=f_i$ for all i. Consider the positive self-adjoint operator $S=T^*T$. By our assumption, there exists a number $M=M(\|S\|)$ and a partition $\{\sigma_j\}_{j=1}^M$ of the set $\{1,\cdots,n\}$ such that

$$\langle SP_{\sigma_j}f, P_{\sigma_j f}\rangle \ge c\|P_{\sigma_j}f\|^2 \quad \text{for all } j \text{ and all } f\in\mathcal{H}_n.$$

Now,

$$\|TP_{\sigma_j}f\|^2 = \langle TP_{\sigma_j}f, TP_{\sigma_j}f\rangle = \langle T^*TP_{\sigma_j}f, P_{\sigma_j}f\rangle = \langle SP_{\sigma_j}f, P_{\sigma_j}f\rangle =$$
$$\langle S^{1/2}P_{\sigma_j}f, S^{1/2}P_{\sigma_j}f\rangle = \|S^{1/2}P_{\sigma_j}f\|^2 \ge c\|P_{\sigma_j}f\|^2.$$

[1]there is nothing special about the constant 4 in the inequalities; it can be replaced by any constant larger than 1 in all the results below.

□

To prove that Kadison-Singer is equivalent to Strong Bourgain-Tzafriri, we will also use a recent result of Weaver [W] (Conjecture KS'_r) giving an equivalent form of the Kadison-Singer Problem in terms of frames.

Theorem 2.2 (Weaver). *The following are equivalent:*

(1) The Kadison-Singer Problem has a positive solution.

(2) There is some natural number r so that there exists universal constants $K \geq 4$ and $\epsilon > \sqrt{K}$ such that the following holds. Let $\{f_i\}_{i=1}^M$ be elements of $\mathcal{H}_N$ satisfying $\|f_i\| = 1$ for all i and suppose

$$\sum_{i=1}^{M} |\langle f, f_i\rangle|^2 \leq K$$

for every unit vector $f \in \mathcal{H}_N$. Then there exists a partition $\{I_j\}_{j=1}^r$ of $\{1, 2, \cdots M\}$ such that

$$\sum_{i\in I_j} |\langle f, f_i\rangle|^2 \leq K - \epsilon$$

for every unit vector $f \in \mathcal{H}_N$ and all $j = 1, 2, \cdots, r$.

For the main theorem we will also need a well known result concerning dual bases.

Proposition 2.3. *Let $\{f_i, f_i^*\}_{i=1}^N$ be vectors in $\mathcal{H}_N$ satisfying $\langle f_i, f_j^*\rangle = \delta_{ij}$ and for all sequences of scalars $\{a_i\}_{i=1}^N$ we have*

$$\Big\| \sum_{i=1}^{N} a_i f_i \Big\| \geq c \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2}.$$

Then for all sequences of scalars $\{b_i\}_{i=1}^N$ we have

$$\Big\| \sum_{i=1}^{N} b_i f_i^* \Big\| \leq \frac{1}{c} \left(\sum_{i=1}^{N} |b_i|^2 \right)^{1/2}.$$

Proof Given scalars $\{b_i\}_{i=1}^N$ we can find scalars $\{a_i\}_{i=1}^N$ so that

$$\Big\| \sum_{i=1}^{N} a_i f_i \Big\| = 1 \geq c \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2},$$

and

$$\Big\| \sum_{i=1}^{N} b_i f_i^* \Big\| = \sum_{i=1}^{N} b_i a_i \leq \left(\sum_{i=1}^{N} |b_i|^2 \right)^{1/2} \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2}$$

$$\leq \frac{1}{c}\Big\| \sum_{i=1}^{N} a_i f_i \Big\| \left(\sum_{i=1}^{M} |b_i|^2 \right)^{1/2} = \frac{1}{c} \left(\sum_{i=1}^{M} |b_i|^2 \right)^{1/2}.$$

□

We are now ready to prove the main result of this section.

Theorem 2.4. *The following are equivalent:*
(1) The Kadison-Singer Problem has a positive solution.
(2) Strong B-T has a positive solution.

Proof. (1) ⇒ (2): Given the validity of the Paving Conjecture, with $\epsilon > 0$ fixed and given a positive self-adjoint $n\times n$-matrix S with ones on the diagonal, there is a partition $\{\sigma_j\}_{j=1}^{M}$ of $\{1, 2, \cdots n\}$ with $M = f(\|S\|)$ so that for every $j = 1, 2, \cdots, M$ we have

$$\|P_{\sigma_j}(S - I)P_{\sigma_j}\| \leq \epsilon.$$

Now, for all $f \in \ell_2^n$ we have

$$\langle SP_{\sigma_j} f, P_{\sigma_j} f\rangle = \langle P_{\sigma_j} SP_{\sigma_j} f, P_{\sigma_j} f\rangle =$$

$$\langle P_{\sigma_j} f, P_{\sigma_j} f\rangle - \langle P_{\sigma_j}(I - S)P_{\sigma_j} f, P_{\sigma_j} f\rangle \geq$$

$$\|P_{\sigma_j} f\|^2 - \|P_{\sigma_j}(I - S)P_{\sigma_j}P_{\sigma_j} f\|\|P_{\sigma_j} f\| \geq$$

$$\|P_{\sigma_j} f\|^2 - \epsilon\|P_{\sigma_j} f\|\|P_{\sigma_j} f\| \geq (1 - \epsilon)\|P_{\sigma_j} f\|^2.$$

This verifies the condition given in Proposition 2.1.

(2) ⇒ (1): We will show that Strong B-T implies the condition of Weaver stated in Theorem 2.2. Choose the universal constant $c > 0$ in Strong B-T. Choose K so that

$$\frac{1}{c^2} \leq K - 2\sqrt{K}.$$

Also, let $\epsilon = 2\sqrt{K}$. Assume that $\{f_i\}_{i=1}^{n}$ is a sequence of norm one vectors in $\mathcal{H}_N$ satisfying

$$\sum_{i=1}^{n} |\langle f, f_i\rangle|^2 \leq K,$$

for every norm one vector $f \in \mathcal{H}_N$. We may assume that $\mathcal{H}_N \subset \ell_2^n$ and define $T : \ell_2^n \to \ell_2^n$ by $Te_i = f_i$. Note that for all $f \in \ell_2^n$ we have

$$\langle T^* f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle.$$

Hence,

$$\|T^* f\|^2 = \sum_{i=1}^{n} |\langle f, f_i\rangle|^2 \leq K\|f\|^2.$$

That is, $\|T^*\| \le \sqrt{K}$. By Strong B-T, there is an $M = f(\|T\|)$ and a partition $\{I_j\}_{j=1}^M$ of $\{1, 2, \cdots, n\}$ so that for j and all sequences of complex numbers $\{a_i\}_{i \in I_j}$ we have:

$$\Big\| \sum_{i \in I_j} a_i f_i \Big\| \ge c \left(\sum_{i \in I_j} |a_i|^2 \right)^{1/2}.$$

It follows that there are elements $\{f_i^*\}_{i=1}^n$ in ℓ_2^n satisfying Proposition 2.3. In particular, $1 \le \|f_i^*\| \le \frac{1}{c}$. We define $T_j^* : \ell_2^n \to \ell_2^n$ by $T_j^* e_i = f_i^*/\|f_i^*\|$ if $i \in I_j$ and $T_j^* e_i = e_i$ if $i \notin I_j$. Now we check the norm of the operator T_j^*. For any sequence of scalars $\{a_i\}_{i=1}^n$ we have

$$\Big\| T_j^* \sum_{i=1}^n a_i e_i \Big\| = \Big\| \sum_{i=1}^n a_i T_j^* e_i \Big\| \le \Big\| \sum_{i \in I_j} a_i \frac{f_i^*}{\|f_i^*\|} \Big\| + \Big\| \sum_{i \notin I_j} a_i e_i \Big\| \le$$

$$\Big\| \sum_{i=1}^n a_i f_i^* \Big\| + \sqrt{\sum_{i \notin I_j} |a_i|^2} \le \frac{1}{c} \sqrt{\sum_{i \in I_j} |a_i|^2} + \sqrt{\sum_{i \notin I_j} |a_i|^2} \le (\frac{1}{c} + 1) \sqrt{\sum_{i=1}^n |a_i|^2},$$

where in the next to last inequality we applied Proposition 2.3. Thus, $\|T_j^*\| \le \frac{1}{c} + 1$. Applying Strong B-T again we can partition I_j into $\{I_{jk}\}_{k=1}^{r_j}$ (where $r(j) = f(\frac{1}{c} + 1)$ is independent of n and N) sets so that for all $k = 1, 2, \cdots, r_j$ and all sequences of scalars $\{a_i\}_{i \in I_{jk}}$,

$$\Big\| \sum_{i \in I_{jk}} a_i f_i^* \Big\| = \Big\| \sum_{i \in I_{jk}} a_i \|f^*\| \frac{f_i^*}{\|f_i^*\|} \Big\|$$

$$\ge c \left(\sum_{i \in I_{jk}} |a_i|^2 \|f_i^*\|^2 \right)^{1/2} \ge c \left(\sum_{i \in I_{jk}} |a_i|^2 \right)^{1/2}.$$

For each j, k let $T_{j,k} = TP_{I_{jk}}$. Again applying Proposition 2.3 we have for all $f = \sum_{i \in I_{jk}} a_i e_i$,

$$\|T_{j,k} f\| = \Big\| \sum_{i \in I_{jk}} a_i f_i \Big\| \le \frac{1}{c} \left(\sum_{i \in I_{jk}} |a_i|^2 \right)^{1/2} = \frac{1}{c} \|f\|.$$

That is $\|T_{j,k}\| \le \frac{1}{c}$. Finally, we have for all norm one vectors $f \in \ell_2^n$ (and hence for all $f \in H_N \subset \ell_2^n$)

$$\sum_{i \in I_{jk}} |\langle f, f_i \rangle|^2 = \|T_{j,k}^* f\|^2 \le \|T_{j,k}^*\|^2 = \|T_{j,k}\|^2 \le \frac{1}{c^2} \le K - 2\sqrt{K}.$$

This establishes the vality of the conditions in Theorem 2.2 with $r = \sum_{j=1}^{M} r_j$ independent of N and n. □

Theorems 2.1 and 2.4 show that the "obvious approach" to proving Strong B-T (i.e. a simple triangle inequality as in the proof of the implication (1) ⇒ (2) in Theorem 2.4) is actually the only approach to proving it. Also, it shows that the universal constant $c > 0$ in Strong B-T used for the lower ℓ_2-bound can actually be replaced by $1 - \epsilon$ for any $\epsilon > 0$ (but in return we have to increase the number of sets we partition $\{1, 2, \cdots, N\}$ into).

Theorem 2.5. *The following are equivalent:*

(1) Strong B-T has a positive solution.

(2) The Paving Conjecture has a positive solution.

(3) The Paving Conjecture has a positive solution for positive self-adjoint operators A with one's on the diagonal.

(4) For every $\epsilon, B > 0$ there is a natural number $M = M(B, \epsilon)$ so that for any natural number n if $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for all $1 \le i \le n$ and $\|T\| \le B$, then there is a partition $\{I_j\}_{j=1}^{M}$ of $\{1, 2, \cdots, n\}$ so that for each $1 \le j \le M$ and all choices of scalars $\{a_i\}_{i \in I_j}$ we have:

$$\Big\| \sum_{i \in I_j} a_i T e_i \Big\|^2 \ge (1 - \epsilon) \sum_{i \in I_j} |a_i|^2.$$

Proof. (1) ⇔ (2): This is Theorem 2.4.

(2) ⇒ (3) and (4) ⇒ (1): These are obvious.

(3) ⇒ (1): This is Proposition 2.1.

(2) ⇒ (4): This is what is actually shown in the proof of Theorem 2.4, (1) ⇒ (2). □

3. Kadison-Singer and Frame Theory

The important point in Strong B-T is that we have to increase the number of elements in our partition as the norm of the operator grows but we always have the same lower ℓ_2-bound c for the partitions. The use of the universal constant c in Theorem 1.2 as a universal lower ℓ_2-bound comes from the form of the proof of the Theorem. We can weaken this statement a little by allowing the lower ℓ_2-bound to depend also on the norm of the linear operator T. This form of the conjecture we will call *Weak B-T.*

Conjecture 3.1 (Weak B-T)**.** *For every $B > 0$ there is a natural number $M = M(B)$ and a $A = A(B) > 0$ so that if $T : \ell_2^n \to \ell_2^n$ is a linear operator for which $\|Te_i\| = 1$, for all $1 \le i \le n$ and $\|T\| \le B$, then there is a partition $\{\sigma_j\}_{j=1}^{M}$ of $\{1, 2, \cdots, n\}$ so that for each $1 \le j \le M$ and all choices of scalars*

$\{a_i\}_{i\in I_j}$ *we have:*

$$\Big\|\sum_{i\in\sigma_j} a_i T e_i\Big\|^2 \geq A \sum_{i\in\sigma_j} |a_i|^2.$$

In [CCLV], the authors show that Weak B-T is equivalent to a conjecture of Feichtinger concerning Hilbert space frames. To state this conjecture we will need some definitions. A family of vectors $\{f_i\}_{i\in I}$ (I may be finite or infinite) of elements of a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ (with *lower frame bround* A and *upper frame bound* B) if for all $f \in \mathcal{H}$ we have:

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

If there is a constant $C > 0$ so that $\|f_i\| \geq C$ for all $i \in I$, we say that the frame is *bounded*. If $\|f_i\| = 1$ for all $i \in I$, we call $\{f_i\}$ a *unit norm frame*.

Recall that $\{f_i\}_{i=1}^{\infty}$ is called a Riesz basic sequence in $\mathcal{H}$ if it is a bounded unconditional basis for its closed linear span. That is, there are constants A, B so that for all sequences of scalars $\{a_i\}_{i=1}^{\infty}$ we have:

$$A\sum_{i=1}^{\infty} |a_i|^2 \leq \|\sum_{i=1}^{\infty} a_i f_i\|^2 \leq B\sum_{i=1}^{\infty} |a_i|^2.$$

The Feichtinger Conjecture states:

Conjecture 3.2 (Feichtinger)**.** *Every bounded frame can be written as a finite union of Riesz basic sequences.*

To relate this to the Kadison-Singer Problem, we need another conjecture from frame theory. But first a definition.

Definition 3.3. *For $\epsilon > 0$, a family $\{f_i\}_{i\in I}$ of vectors in a Hilbert space $\mathcal{H}$ with $\|f_i\| = 1$ is called an ϵ-Riesz basic sequence if for all sets of scalars $\{a_i\}_{i\in I}$ we have:*

$$(1-\epsilon)\sum_{i\in I} |a_i|^2 \leq \Big\|\sum_{i\in I} a_i f_i\Big\|^2 \leq (1+\epsilon)\sum_{i\in I} |a_i|^2.$$

Now we state another conjecture from frame theory.

Conjecture 3.4 (R_ϵ-Conjecture)**.** *For every $\epsilon > 0$, every Riesz basic sequence is a finite union of ϵ-Riesz basic sequences.*

We will also need Proposition 2.1 from [CCLV] which we state here for completeness.

Proposition 3.5. *Fix a natural number M and assume for every natural number n we have a partition $\{I_i^n\}_{i=1}^M$ of $\{1, 2, \cdots, n\}$. Then there are natural numbers $\{n_1 < n_2 < \cdots\}$ so that if $j \in I_i^{n_j}$ for some $i \in \{1, \cdots, M\}$, then $j \in I_i^{n_k}$, for all $k \geq j$. Hence, if $I_i = \{j | j \in I_i^{n_j}\}$ then*

(1) $\{I_i\}_{i=1}^M$ *is a partition of* $\mathbb{N}$.

(2) If $I_i = \{j_1 < j_2 < \cdots\}$ *then for every natural number* k *we have* $\{j_1, j_2, \cdots, j_k\} \subset I_i^{n_{j_k}}$.

Now we can relate our various conjectures.

Theorem 3.6. *The following are equivalent:*

(1) A positive solution to the Kadison-Singer Problem.

(2) A positive solution to the Feichtinger Conjecture and a positive solution to the R_ϵ*-Conjecture.*

(3) For every $\epsilon > 0$*, every bounded frame is a finite union of* ϵ*-Riesz basic sequences.*

(4) For every $\epsilon, B > 0$ *there is a natural number* $M = M(B, \epsilon)$ *so that for every natural number* $n \in \mathbb{N}$ *and for every linear operator* $T : \ell_2^n \to \ell_2^n$ *with* $\|Te_i\| = 1$*, for all* $1 \le i \le n$ *and* $\|T\| \le B$*, there is a partition* $\{I_j\}_{j=1}^M$ *of* $\{1, 2, \cdots, n\}$ *so that for all* $1 \le j \le M$ *and all choices of scalars* $\{a_i\}_{i \in I_j}$ *we have*

$$(1-\epsilon)\sum_{i\in I_j}|a_i|^2 \le \Big\|\sum_{i\in I_j} a_i T e_i\Big\|^2 \le (1+\epsilon)\sum_{i\in I_j}|a_i|^2.$$

Proof. (3) $\Leftrightarrow$ (2): This is obvious.

(4) $\Rightarrow$ (1): This is Theorem 2.5.

(1) $\Rightarrow$ (4): Fix $n, \epsilon, B > 0$ and let $T : \ell_2^n \to \ell_2^n$ be a linear operator with $\|Te_i\| = 1$ for all $1 \le i \le n$ and $\|T\| \le B$. Choose a $0 < \delta < \epsilon$ so that $\frac{1}{1-\delta} \le 1 + \epsilon$. By Theorem 2.5 (4), there is a partition $\{I_j\}_{j=1}^M$ of $\{1, 2, \cdots, n\}$ so that for all $1 \le j \le M$ and for all choices of scalars $\{a_i\}_{i\in I_j}$ we have

$$(1-\delta)\sum_{i\in I_j}|a_i|^2 \le \Big\|\sum_{i\in I_j} a_i T e_i\Big\|^2.$$

It follows that $\{Te_i\}_{i\in I_j}$ is a Riesz basic sequence and hence there is a dual basic sequence $\{f_k\}_{k\in I_j}$ satisfying: $f_k T e_i = \delta_{ki}$. By Proposition 2.3, for all choices of scalars $\{a_i\}_{i\in I_j}$ we have:

$$\Big\|\sum_{i\in I_j} a_i f_i\Big\|^2 \le \frac{1}{1-\delta}\sum_{i\in I_j}|a_i|^2.$$

In particular,

$$1 = \delta_{ii}^2 = [f_i(Te_i)]^2 \le \|f_i\|^2 \le \frac{1}{1-\delta}.$$

Let $g_i = \frac{f_i}{\|f_i\|}$ and $L : \ell_2^n \to \ell_2^n$ be given by $Le_i = g_i$, so that $\|Le_i\| = 1$. Now, for all choices of scalars $\{a_i\}_{i\in I_j}$ we have

$$\Big\|L\sum_{i\in I_j} a_i e_i\Big\|^2 = \Big\|\sum_{i\in I_j} a_i g_i\Big\|^2 = \Big\|\sum_{i\in I_j}\frac{a_i}{\|f_i\|} f_i\Big\|^2 \le \frac{1}{1-\delta}\sum_{i\in I_j}\frac{|a_i|^2}{\|f_i\|^2} \le \frac{1}{1-\delta}\sum_{i\in I_j}|a_i|^2.$$

Hence, $\|L\|^2 \leq \frac{1}{1-\delta}$. Applying Theorem 2.5 (4) again to partition I_j into $M_j = M_j(\frac{1}{(1-\delta)}, \delta)$-sets $\{I_{\ell j}\}_{\ell=1}^{M_j}$ so that for all choices of scalars $\{a_i\}_{i\in I_{\ell j}}$ we have (by our choice of δ)

$$(1-\epsilon)\sum_{i\in I_{\ell j}}|a_i|^2 \leq (1-\delta)\sum_{i\in I_{\ell j}}|a_i|^2 \leq \Big\|\sum_{i\in I_{\ell j}}a_i f_i\Big\|^2$$
$$\leq \frac{1}{1-\delta}\sum_{i\in I_{\ell j}}|a_i|^2 \leq (1+\epsilon)\sum_{i\in I_{\ell j}}|a_i|^2.$$

$(1,4)\Rightarrow(2)$: Proposition 3.1 of [CCLV] shows that a positive solution to the Kadison-Singer Problem implies a positive solution to the Feichtinger Conjecture. So we check the R_ϵ-Conjecture. Let $\{f_i\}_{i\in I}$ be a Riesz basic sequence in ℓ_2 with $\|f_i\| = 1$ and fix $\epsilon > 0$. By (4) of Theorem 2.5, for each natural number n, there is a partition $\{I_j^n\}_{j=1}^M$ of $\{1,2,\cdots,n\}$ so that $\{\varphi_i\}_{i\in I_j^n}$ is an ϵ-Riesz basic sequence for all $j = 1,2,\cdots,M$. Let $\{I_j\}_{j=1}^M$ be the family given in Proposition 3.5. Fix $1 \leq j \leq M$. If $I_j = \{j_1 < j_2 < \cdots\}$, then for every k we have that $\{j_1, j_2,\cdots,j_k\} \subset I_j^{n_k}$ and so $\{f_{j_\ell}\}_{\ell=1}^k$ is a ϵ-Riesz basic sequence. Hence, $\{f_i\}_{i\in I_j}$ is an ϵ-Riesz basic sequence for all $j = 1,2,\cdots,M$.

$(2)\Rightarrow(3)$: This is obvious.

$(3)\Rightarrow(4)$: We proceed by way of contradiction. If (3) holds but (4) fails, then there is an $\epsilon, B > 0$ and natural numbers $m_1 < m_2 < \cdots$ and linear operators $T_n : \ell_2^{m_n} \to \ell_2^{m_n}$ with $\|T_n e_i^n\| = 1$, $\|T_n\| \leq B$ and for every partition $\{I_j^n\}_{j=1}^n$ of $\{1,2,\cdots,m_n\}$ we have that the inequality in (4) fails. Let

$$\mathcal{H} = \left(\sum_{n=1}^\infty \oplus \ell_2^{m_n}\right)_{\ell_2}.$$

Now,

$$\{e_i^n\}_{i=1,n=1}^{m_n,\ \infty} \cup \{T_n e_i^n\}_{i=1,n=1}^{m_n,\ \infty},$$

is a unit norm frame for $\mathcal{H}$ and so by (3) it is a finite union of ϵ-Riesz basic sequences. In particular, $\{T_n e_i^n\}_{i=1,n=1}^{m_n,\ \infty}$ is a union of, say M, ϵ-Riesz basic sequences. For $n > M$, $\{T_n e_i^n\}_{i=1}^{m_n}$ is a union of M δ-Riesz basic sequences say $\{I_j\}_{j=1}^M$. So for every j and all sets of scalars $\{a_i\}_{i\in I_j}$ we have

$$(1-\epsilon)\sum_{i\in I_j}|a_i|^2 \leq \|\sum_{i\in I_j}a_i T_n e_i^n\|^2 \leq (1+\epsilon)|\sum_{i\in I_j}|a_i|^2.$$

This completes the proof of $(3)\Rightarrow(4)$ and of the theorem. □

4. Random Restricted Invertibility

Here we study the the family $\Sigma(T)$ of all sets of the isomorphism of arbitrary bounded linear operator T on ℓ_2.

Our first impression is that the family $\Sigma(T)$ is small. Indeed, for an even integer n, consider the linear operator T on ℓ_2^n defined by $Te_i = \frac{1}{\sqrt{2}} e_{\lceil i/2 \rceil}$. Every subset $\sigma \in \Sigma(T)$ contains no pairs of the form $\{2i-1, 2i\}$, hence $|\Sigma(T)| = 3^{n/2-1} \ll 2^n$. However, the subsets $\{1, 3, 5, \cdots, n-1\}$ and $\{2, 4, 6, \cdots, n\}$ both belong to $\Sigma(T)$, and the average of the characteristic functions of these subsets is a half of the characteristic function of the whole interval $\{1, 2, \cdots, n\}$.

This shows that Bourgain-Tzafriri Restricted-Invertibility Theorem 1.2 does *not* hold for random subsets σ in the classical sence of randomness. However $\Sigma(T)$ is large in another sense – the following result shows that the average of the characteristic functions of the sets $\sigma \in \Sigma(T)$ is nicely bounded below. This implies Bourgain-Tzafriri Theorem, as we will see shortly.

Theorem 4.1. *Let T be a norm one linear operator on ℓ_2. Then there exists a probability measure ν on $\Sigma(T)$ such that*

$$\nu\{\sigma |\, i \in \sigma\} \geq c\|Te_i\|^2 \quad \text{for all } i. \tag{4.1}$$

In this theorem and throughout this section, c denotes a positive absolute constant.

Note that the left hand side of (4.1) clearly equals $\int_{\Sigma(T)} \chi_\sigma(i)\, d\nu(\sigma)$.

The next two results follow immediately from Theorem 3.1. However, as we will see, they also imply Theorem 3.1 and so we will actually prove them independently later and use them to prove Theorem 3.1.

Let μ be a measure on $\mathbb{N}$; for simplicity we will denote $\mu(\{i\})$ by $\mu(i)$. Summing over i with weights $\mu(i)$ in (4.1), we obtain

$$\int_{\Sigma(T)} \mu(\sigma)\, d\nu(\sigma) = \sum_i \mu(i) \int_{\Sigma(T)} \chi_\sigma(i)\, d\nu(\sigma) \geq c \sum_i \mu(i) \|Te_i\|^2.$$

This proves the following corollary.

Corollary 4.2. *Let T be a norm one linear operator on ℓ_2, and let μ be a measure on $\mathbb{N}$. Then there exists a set $\sigma \in \Sigma(T)$ such that*

$$\mu(\sigma) \geq c \sum_i \mu(i) \|Te_i\|^2. \tag{4.2}$$

Corollary 3.2 was essentially proved by S.Szarek [Sz] with only the *upper* bound in (1.1).

For the counting measure on $\mathbb{N}$, Corollary 4.2 proves the existence of a set $\sigma \in \Sigma(T)$ with cardinality $|\sigma| \geq c\|T\|_{\mathrm{HS}}^2$ (where $\|T\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of T). It is known [V] that the constant c in this estimate can be improved to $1 - \varepsilon$ for any $\varepsilon > 0$ at the cost of replacing 4 in the definition of $\Sigma(T)$ by a number depending on ε.

An immediate consequence of Corollary 4.2 is

Corollary 4.3. *Let T be a linear operator on ℓ_2 with $\|Te_i\| = 1$ for all i, and let μ be a measure on $\mathbb{N}$. Then there exits a subset σ of the integers such that $\mu(\sigma) \geq c/\|T\|^2$ and*

$$\frac{1}{2}\|x\| \leq \|Tx\| \leq 2\|x\| \quad \textit{for } x \in \mathbb{R}^\sigma. \tag{4.3}$$

This result contains the Bourgain-Tzafriri restricted-invertibility principle, Theorem 1.2, as is seen by considering the uniform measure on $\{1, \cdots, n\}$.

We pass now to the proof of Theorem 4.1, which relies on the methods of Bourgain-Tzafriri [B-Tz] and of [V].

In [B-Tz], a suppression analogue of Theorem 4.1 is proved. By P_σ we denote the orthogonal projection in $\mathbb{R}^n$ onto $\mathbb{R}^\sigma$, where σ is a subset of $\{1, \cdots, n\}$.

Theorem 4.4 (Bourgain-Tzafriri)**.** *Let S be a linear operator on ℓ_2 whose matrix relative to the unit vector basis has zero diagonal. For an $\varepsilon > 0$, denote by $\Sigma'(S, \varepsilon)$ the family of all subsets σ of the integers such that $\|P_\sigma S P_\sigma\| \leq \varepsilon\|S\|$. Then there exists a probability measure ν' on $\Sigma'(S, \varepsilon)$ such that*

$$\nu'\{\sigma | \, i \in \sigma\} \geq c\varepsilon^2 \quad \textit{for all } i. \tag{4.4}$$

This implies a weaker version of Corollary 4.3. Indeed, under the assumptions of Corollary 4.3 we automatically have $\|T\| \geq 1$; let $S = T^*T - I$ and $\varepsilon = \frac{1}{2\|S\|} \geq \frac{1}{4\|T\|^2}$. Then for every $\sigma \in \Sigma'(S, \varepsilon)$ and every $x \in \mathbb{R}^\sigma$, $\|x\| = 1$, we have

$$\big| \|Tx\|^2 - \|x\|^2 \big| = \langle Sx, x \rangle = \langle P_\sigma S P_\sigma x, x \rangle \leq \varepsilon \|S\| = \frac{1}{2}.$$

Hence (4.3) holds. Now, summing over i in (4.4), we obtain

$$\begin{aligned} \int_{\Sigma'(S,\varepsilon)} \mu(\sigma) \, d\nu'(\sigma) &= \sum_i \mu(i) \int_{\Sigma'(S,\varepsilon)} \chi_\sigma(i) \, d\nu'(\sigma) \\ &\geq \Big(\sum_i \mu(i) \Big) c\varepsilon^2 \gtrsim 1/\|T\|^4. \end{aligned}$$

Replacing the integral by the maximum, we conclude:

$$\text{Corollary 4.3 holds with } \mu(\sigma) \geq c/\|T\|^4. \tag{4.5}$$

We will use this weaker estimate in the proof of the actual Corollary 4.3. One way to do this is to apply the weaker form of Corollary 4.2 due to Szarek (see the remark above) and then apply (4.5) to the operator $T_1 : \ell_2^\sigma \to \ell_2$ that sends e_i to $\frac{Te_i}{\|Te_i\|}$, $i \in \sigma$. Since $\|T_1\| \leq 2$, Corollary 4.3 will follow.

We will chose another way, which will actually yield a stronger form of Szarek's result (i.e. with both sides in (1.1)) as a byproduct. The proof of Theorem 4.1 will be organized backwards: Corollary 4.3 $\Longrightarrow$ Corollary 4.2 $\Longrightarrow$ Theorem 4.1. Corollary 4.3 itself will be a consequence of the following suppression result, which is a "weighted" variant of an unpublished theorem of Kashin and Tzafriri [K-Tz] (see [V]) and a slight improvement of a result of Szarek [Sz].

Theorem 4.5. *Let T be a linear operator on ℓ_2, and μ be a probability measure on $\mathbb{N}$. Then for any $0 < \delta < 1/4$ there exists a subset σ of the integers with $\mu(\sigma) \geq \delta$ and such that*

$$\|TP_\sigma\| \leq c\sqrt{\delta}\|T\| + c\Big(\sum_i \mu(i)\|Te_i\|^2\Big)^{1/2}.$$

The proof uses standard tools, a random selection followed by the Grothendieck factorization. The random selection is done in the next lemma. Let $0 < \delta < 1$. Consider random selectors δ_i, i.e. independent $\{0,1\}$-valued random variables with $\mathbb{E}\delta_i = \delta$. Then the linear operator $P_\delta := \sum_i \delta_i\, e_i \otimes e_i$ is a random orthogonal projection in ℓ_2.

Lemma 4.6. *Let T be a linear operator on ℓ_2^n and let μ be a probability measure on $\{1, \cdots, n\}$. Then for $0 < \delta < 1/4$ a random coordinate projection $P_\delta := \sum_{i=1}^n \delta_i\, e_i \otimes e_i$ satisfies*

$$\mathbb{E}\|P_\delta T^*\|_{\ell_2^n \to L_1^n(\sqrt{\mu})} \leq \delta\|T\| + 2\sqrt{\delta}\Big(\sum_{i=1}^n \mu(i)\|Te_i\|^2\Big)^{1/2},$$

where the space $L_1^n(\sqrt{\mu})$ is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_{L_1^n(\sqrt{\mu})} = \sum_{i=1}^n \sqrt{\mu(i)}|x(i)|.$$

Proof. This is the Gine-Zinn's symmetrization scheme,

$$\mathbb{E}\|P_\delta T^*\|_{\ell_2^n \to L_1^n(\sqrt{\mu})} = \mathbb{E} \sup_{x \in B(\ell_2^n)} \sum_{i=1}^n \delta_i \sqrt{\mu(i)}|\langle Te_i, x\rangle| \tag{4.6}$$

$$\leq \delta \sup_{x \in B(\ell_2^n)} \sum_{i=1}^n \sqrt{\mu(i)}|\langle Te_i, x\rangle| + \mathbb{E} \sup_{x \in B(\ell_2^n)} \sum_{i=1}^n (\delta_i - \delta)\sqrt{\mu(i)}|\langle Te_i, x\rangle|.$$

By Hölder's inequality, the first sumand is bounded by

$$\delta \sup_{x \in B(\ell_2^n)} \Big(\sum_{i=1}^n |\langle Te_i, x\rangle|^2\Big)^{1/2} = \delta\|T\|.$$

To bound the second summand in (4.6), let δ_i' be independent copies of δ_i. Then $(\delta_i - \delta)$ can be replaced by $(\delta_i - \delta_i')$, which (by the symmetry) has the same distribution as $\varepsilon_i(\delta_i - \delta_i')$, where ε_i denote Rademacher random variables (independent random variables taking values -1 and 1 with probability 1/2). So $(\delta_i - \delta)$ in (4.6) can be replaced by $2\varepsilon_i\delta_i$, which can further be replaced (by the standard comparison inequality) by $g_i\delta_i$, where g_i are independent normalized Gaussian random variables. These probabilistic techniques, as well as Slepian's inequality below, can be found in [L-T] sections 3 and 6. Hence

$$\mathbb{E}\|P_\delta T^*\|_{\ell_2^n \to L_1^n(\sqrt{\mu})} \le \delta\|T\| + 2\mathbb{E} \sup_{x \in B(\ell_2^n)} \sum_{i=1}^n g_i \delta_i \sqrt{\mu(i)} |\langle Te_i, x\rangle|.$$

By Slepian's inequality, $|\langle Te_i, x\rangle|$ can be replaced by $\langle Te_i, x\rangle$, and we continue the estimate as

$$\begin{aligned} &\le \delta\|T\| + 2\mathbb{E}\Big\| \sum_{i=1}^n g_i \delta_i \sqrt{\mu(i)} Te_i \Big\| \\ &\le \delta\|T\| + 2\Big(\mathbb{E} \sum_{i=1}^n \delta_i \mu(i) \|Te_i\|^2\Big)^{1/2} \\ &= \delta\|T\| + 2\sqrt{\delta}\Big(\sum_{i=1}^n \mu(i)\|Te_i\|^2\Big)^{1/2}. \end{aligned}$$

The proof is complete. ■

The Grothendieck factorization is done in the following lemma.

Lemma 4.7. *Let $T : \ell_2^n \to \ell_2$ be a linear operator and let μ be a measure on $\{1, \cdots, n\}$ of total mass m. Then there exists a subset $\sigma \in \{1, \cdots, n\}$ such that $\mu(\sigma) \ge m/2$ and*

$$\|TP_\sigma\|_{\ell_2^n \to \ell_2} \le \frac{c}{\sqrt{m}} \|T\|_{L_\infty^n(\sqrt{\mu}) \to \ell_2},$$

where the space $L_\infty^n(\sqrt{\mu})$ is $\mathbb{R}^n$ equipped with the norm

$$\|x\|_{L_\infty^n(\sqrt{\mu})} = \max_{i \le n} \frac{|x(i)|}{\sqrt{\mu(i)}}.$$

Proof. Consider the isometry $\Delta : L_\infty^n(\sqrt{\mu}) \to L_\infty^n$ defined as $(\Delta x)(i) = \frac{|x(i)|}{\sqrt{\mu(i)}}$. By the Grothendieck theorem (see [TJ] Corollary 10.10), the operator $T\Delta^{-1} : L_\infty^n \to \ell_2$ is 2-summing, and its 2-summing norm is bounded as

$$\pi_2(T\Delta^{-1}) \le c\|T\Delta^{-1}\|_{L_\infty^n \to \ell_2} = c\|T\|_{L_\infty^n(\sqrt{\mu}) \to \ell_2} =: M.$$

By Pietsch's Theorem (see [TJ] Corollary 9.4), there exists a probability measure λ on $[n] = \{1, \cdots, n\}$ so that

$$\|T\Delta^{-1}\|_{L_2([n],\lambda) \to \ell_2} \le M.$$

Hence for every $x \in \mathbb{R}^n$,

$$\|T\Delta^{-1}x\|_{\ell_2} \le M\Big(\sum_{i=1}^n \lambda(i)|x(i)|^2\Big)^{1/2},$$

and thus

$$\|Tx\|_{\ell_2} \le M\Big(\sum_{i=1}^{n} \frac{\lambda(i)}{\mu(i)}|x(i)|^2\Big)^{1/2}.$$

Since $\frac{1}{m}\int_{[n]} \frac{\lambda(i)}{\mu(i)}\, d\mu(i) = \frac{1}{m}\sum_{i=1}^{n} \lambda(i) = \frac{1}{m}$, Chebyshev's inequality gives

$$\frac{1}{m}\mu\Big\{i \;\Big|\; \frac{\lambda(i)}{\mu(i)} \le \frac{2}{m}\Big\} \ge 1/2.$$

Hence there exists a subset $\sigma \subset [n]$ with $\mu(\sigma) \ge m/2$ and such that

$$\frac{\lambda(i)}{\mu(i)} \le \frac{2}{m} \quad \text{for all } i \in \sigma.$$

Thus for every $x \in \mathbb{R}^\sigma$

$$\|Tx\|_{\ell_2} \le \sqrt{\frac{2}{m}} M\Big(\sum_{i=1}^{n} |x(i)|^2\Big)^{1/2}.$$

This completes the proof. ■

Proof of Theorem 4.5. By approximation, we can assume that T acts on ℓ_2^n and $\delta > C/n$ with a sufficiently large absolute constant C. Then we apply Lemma 4.6. Since $\mathbb{E}\mu(\sigma) = \mathbb{E}\sum_{i=1}^{n} \delta_i \mu(i) = \delta$, the classical estimates on the binomial tails imply that $\mu(\sigma) \ge \delta/2$ with probability greater than $1/2$. Then there exists a subset $\sigma \subset \{1, \cdots, n\}$ such that $\mu(\sigma) \ge \delta/2$ and, by duality,

$$\|TP_\sigma\|_{L_\infty^n(\sqrt{\mu}) \to \ell_2^n} \le 2\delta\|T\| + 4\sqrt{\delta}\Big(\sum_{i=1}^{n} \mu(i)\|Te_i\|^2\Big)^{1/2}.$$

Next we apply Lemma 4.7 for the operator $TP_\sigma : L_\infty^\sigma(\sqrt{\mu}) \to \ell_2^n$ with $m = \mu(\sigma) \ge \delta/2$. There exists a subset $\sigma' \subset \sigma$ with $\mu(\sigma') \ge \delta/4$ and such that

$$\begin{aligned}\|TP_{\sigma'}\|_{\ell_2^\sigma \to \ell_2^n} &\le \frac{c}{\sqrt{\delta}}\|T\|_{L_\infty^\sigma(\sqrt{\mu}) \to \ell_2^n} \\ &\le c\sqrt{\delta}\|T\| + c\Big(\sum_i \mu(i)\|Te_i\|^2\Big)^{1/2}.\end{aligned}$$

This proves the theorem. ■

Proof of Corollary 4.3. Applying Theorem 4.5 with $\delta = \frac{1}{4\|T\|^2}$, we find a subset $\sigma \subset \mathbb{N}$ with $\mu(\sigma) \ge \frac{1}{4\|T\|^2}$ and such that

$$\|TP_\sigma\| \le c.$$

Next, we apply Theorem 4.4; more precisely, its consequence (4.5) for the operator $TP_\sigma : \ell_2 \to \ell_2$ and for the probability measure μ conditioned on σ, i.e. for μ' defined as $\mu'(\eta) = \mu(\eta \cap \sigma)/\mu(\sigma)$, $\eta \subset \mathbb{N}$. There exists a subset $\sigma' \subset \sigma$ with $\mu(\sigma') \geq c\mu(\sigma) \geq c/\|T\|^2$ and such that (4.3) holds for all $x \in \mathbb{R}^{\sigma'}$. This completes the proof. ■

To prove Corollary 4.2, we introduce a splitting procedure. Consider a family η_k, $k = 1, \cdots, n$ of disjoint subsets of the integers. This family defines a splitting of a probability measure λ on $\{1, \cdots, n\}$ and of any sequence $(x_i)_{i \leq n}$ in ℓ_2. Namely, put

$$\eta = \bigcup_{i \leq n} \eta_i, \quad N = |\eta|, \quad \text{and } N_i = |\eta_i|.$$

Then the splitted probability measure λ' on η and the splitted sequence $(x'_k)_{k \in \eta}$ are defined as

$$\lambda'(k) = \frac{\lambda(i)}{N_i}, \quad x'_k = \frac{x_i}{\sqrt{N_i}} \quad \text{for } k \in \eta_i.$$

Splitting will be used to make the norms $\|x_i\|$ almost identical. Namely, one can easily construct a splitting such that

$$0.9\|x'_l\| \leq \|x'_k\| \leq 1.1\|x'_l\| \quad \text{for all } k, l \in \eta.$$

Since $\sum_{k \in \eta} \|x'_k\|^2 = \sum_{i=1}^n \|x_i\|^2 =: h$, we have a postiori:

$$\|x'_k\| \sim \sqrt{\frac{h}{N}} \quad \text{for all } k \in \eta,$$

where $a \sim b$ means $\frac{1}{2}a \leq b \leq 2a$. Moreover, since $\|x'_k\| = \frac{\|x_i\|}{\sqrt{N_i}}$ for $k \in \eta_i$, we also have

$$\frac{1}{N_i} = \frac{\|x'_k\|^2}{\|x_i\|^2} \sim \frac{h}{N} \cdot \frac{1}{\|x_i\|^2},$$

hence

$$\lambda'(k) = \frac{\lambda(i)}{N_i} \sim \frac{h}{N} \cdot \frac{\lambda(i)}{\|x_i\|^2} \quad \text{for } k \in \eta_i.$$

Let $T : \ell_2^n \to \ell_2$ be a linear operator defined as $Te_i = x_i$, $i = 1, \cdots, n$. The splitting of T is defined as the linear operator $T' : \ell_2^\eta \to \ell_2$ acting as $T'e_k = x'_k$, $k \in \eta$. An easily checked but important property is

$$\|T'\| \leq \|T\|.$$

Proof of Corollary 4.2. By approximation, we can assume that T is an operator from ℓ_2^n into ℓ_2. Then, by the procedure described above, for any probability measure λ on $\{1, \cdots, n\}$ there exists a splitting $\eta = \bigcup_{i \leq n} \eta_i$, $|\eta| = N$, such that that the splitted measure λ' on η and the splitted operator $T' : \ell_2^\eta \to \ell_2$ satisfy:

(1) $\|T'\| \leq \|T\| \leq 1$,

(2) For $k \in \eta$, $\|T'e_k\| \sim \sqrt{\frac{h}{N}}$, where $h = \|T\|_{\mathrm{HS}}^2$,
(3) For $k \in \eta_i$, $\lambda'(k) \sim \frac{h}{N} \cdot \frac{\lambda(i)}{\|Te_i\|^2}$.

We apply Corollary 4.3 to the operator $S : \ell_2^\eta \to \ell_2$ defined as

$$Se_k = \frac{T'e_k}{\|T'e_k\|}, \quad k \in \eta.$$

Note that

$$\|S\| \le \max_{k \in \eta} \frac{\|T\|}{\|T'e_k\|} \le 2\sqrt{\frac{N}{h}}.$$

Therefore there exists a subset $\sigma' \subset \eta$ such that

$$\sigma' \in K(S) \quad \text{and} \quad \lambda'(\sigma') \ge c\frac{h}{N}. \tag{4.7}$$

Now the crucial fact is that for every i the set $\sigma' \cap \eta_i$ contains at most one element (denoted by k_i if it exists). This is because for a fixed i, the vectors $(Se_k, k \in \eta_i)$ are all multiples of one vector; while, since $\sigma' \in K(S)$, the set $(Se_k, k \in \sigma')$ can not contain colinear vectors, as they would fail the lower bound in (1.1).

Let $\sigma = \{i \mid \sigma' \cap \eta_i \ne \emptyset\}$. Then $\sigma \in \Sigma(T)$, and

$$\lambda'(\sigma') = \sum_{i \in \sigma} \lambda'(k_i) \sim \frac{h}{N} \sum_{i \in \sigma} \frac{\lambda(i)}{\|Te_i\|^2}.$$

This and (4.7) imply

$$\sum_{i \in \sigma} \frac{\lambda(i)}{\|Te_i\|^2} \ge c. \tag{4.8}$$

The conclusion of the Corollary follows by applying (4.8) to the probability measure λ defined as

$$\lambda(i) = \frac{\mu(i)\|Te_i\|^2}{\sum_{i=1}^n \mu(i)\|Te_i\|^2}, \quad i = 1, \cdots, n.$$

■

Actually, (4.2) and (4.8) are easily seen to be equivalent. Indeed, one can get (4.8) by applying Corollary 4.2 to the measure μ defined by $\mu(i) = \frac{\lambda(i)}{\|Te_i\|^2}$.

Proof of Theorem 4.1. This argument is a minor adaptation of [B-Tz] Corolary 1.4. $\Sigma(T)$ is a w^*-compact set. For each integer i, define a function $\pi_i \in C(\Sigma(T))$ by setting

$$\pi_i(\sigma) = \frac{\chi_\sigma(i)}{\|Te_i\|^2}, \quad \sigma \in \Sigma(T).$$

Let H be the convex hull of the functions (π_i). Fix a $\pi \in H$ and write it as a convex combination $\pi = \sum_i \lambda_i \pi_i$. By Corollary 4.2, or rather by its consequence (4.8), there exists a set $\sigma \in \Sigma(T)$ such that $\pi(\sigma) \geq c$. Looking at σ as a point evaluation functional on $C(K)$, we conclude by the Hahn-Banach theorem that there exists a probability measure $\nu \in C(\Sigma(T))^*$ such that

$$\nu(\pi) = \int_{\Sigma(T)} \pi(\sigma)\, d\nu(\sigma) \geq c \quad \text{for all } \pi \in H.$$

Applying this estimate for $\pi = \pi_i$, we obtain

$$\int_{\Sigma(T)} \chi_\sigma(i)\, d\nu(\sigma) \geq c\|Te_i\|^2,$$

which is exactly the conclusion of the theorem. ■

While Theorem 4.1 seems to be unable to yield the conjectures under discussion, it proves that every bounded frame (and actually very bounded Bessel sequence) has *many* Riesz basic subsequences.

Theorem 4.8. *If $\{f_i\}_{i=1}^\infty$ is a Bessel sequence with Bessel constant $B > 0$ and $\|f_i\| = 1$, then there exists a probability measure ν on the set K of all Riesz basic subsequences of $\{f_i\}$ with Riesz basis constant 2, such that the measure ν of the subsequences in K that contain any given element f_i is at least $b = b(B) > 0$.*

Finally, we state a conjectured "paving" analogue of Theorem 4.1 that would clearly imply Conjectures 3.2, 3.1, 1.3 and 1.1.

Conjecture 4.9. *Let T be a linear operator on ℓ_2^n such that $\|Te_i\| = 1$ for all i. Then there exists a partition $\{\sigma_k\}_{k \leq M}$ of the set $\{1, \cdots, n\}$, where M depends only on the norm of T, and such that $\sigma_k \in \Sigma(T)$ for all k.*

To see the relation to Theorem 4.1, assume that this conjecture is true, and let ν be the probability measure on $\Sigma(T)$ that assigns each $\sigma(k)$ measure $1/M$. Then

$$\nu\{\sigma |\, i \in \sigma\} \geq 1/M \quad \text{for all } i.$$

If moreover $M \sim 1/\|T\|^2$, then this clearly implies Theorem 4.1 for operators T such that $\|Te_i\|$ have same value for all i.

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