**KALTON AND QUASI-BANACH SPACES**

Nigel Kalton's general solution to the three-space problem [3, 4] was the first piece of serious research mathematics I got to read as an undergraduate student in 1995. The elegance of the construction of the Kalton-Peck's space $Z_2$ and the technical power of those influential papers struck me then. Nigel became my first mathematical hero, and four years later, my Ph.D. advisor in Columbia, Missouri. Ironically, by that time our interests had somewhat diverged as I had become more interested in geometric functional analysis. Although as a result we have not really collaborated, Nigel has remained to me what he had been in the first place, a master. Elegance and insight are the best words to describe his mathematics.

Among many areas of analysis Nigel worked in, the geometry of *quasi-Banach spaces* occupied a lot of his attention [6]. What differs quasi-Banach spaces from the more classical Banach spaces is the triangle inequality. In quasi-Banach spaces, the triangle inequality is allowed to hold approximately:

$$\|x+y\| \leq C(\|x\|+\|y\|)$$

for some constant $C \geq 1$. This relaxation leads to a broad class of spaces. Lebesgue spaces $L_p$ and Hardy spaces $H_p$ are Banach spaces for $1 < p \leq \infty$ and only quasi-Banach spaces for $0 < p < 1$. In fact the topology of every locally bounded topological vector space $X$ can be induced by a suitable quasi-norm.

Because of the weakness of the triangle inequality, convexity arguments do not work well in quasi-Banach spaces (although the *Aoki-Rolewicz theorem* offers a remarkable surrogate, an equivalent quasi-norm $|\cdot|$ on $X$ which satisfies $|x+y|^p \leq |x|^p + |y|^p$ for some $0 < p < 1$). Anyway, the absence of genuine convexity is a stumbling block that can make simple-looking problems difficult. For example, the *atomic space problem* is still open, which asks whether every quasi-Banach space has a proper closed infinite-dimensional subspace. The most basic result of Banach space theory, the Hahn-Banach extension theorem, was shown by Kalton to fail in *all* quasi-Banach spaces that are not Banach spaces [2].

Kalton discovered surprising ways to put quasi-Banach spaces into action for problems about usual Banach spaces. One of the central questions in the field, the *three-space problem*, is about short exact sequences of Banach spaces

$$0 \to X \to Z \to Y \to 0.$$

Such a space $Z$ is called a *twisted sum* of the spaces $X$ and $Y$. The three-space problem asks: what properties of $X$ and $Y$ are inherited by their twisted sums $Z$ [5]? For example, must $Z$ be isomorphic to a Hilbert space whenever both $X$ and $Y$ are?

The latter question was answered negatively by Enflo, Lindenstrauss and Pisier in 1975 [1]. Shortly after that, Kalton and Peck developed an elegant approach to this problem and to twisted sums in general, which was based on quasi-Banach spaces [4]. A remarkable product of this theory is the Kalton-Peck's space $Z_p$ that is not isomorphic to $\ell_p$ while being a twisted sum of $X = Y = \ell_p$, where $1 < p < \infty$.

The hidden link between twisted sums and quasi-Banach spaces is the notion of *quasi-linear map*, which is a homogeneous transformation $F : Y \to X$ that satisfies

$$\|F(y_1 + y_2) - F(y_1) - F(y_2)\| \le K(\|y_1\| + \|y_2\|)$$

with a suitable constant $K$. Quasi-linear maps can be easily constructed from twisted sums: the difference between a bounded homogeneous and a linear selection of the quotient map $Z \to Y$ is a quasi-linear map. Conversely, twisted sums can be constructed from quasi-linear maps: one can take the Cartesian product $Z := X \times Y$ with the quasi-norm

$$\|(x, y)\| := \|x - F(y)\| + \|y\|.$$

A trivial choice of $F = 0$ and, more generally, of a linear map $F$, produces a twisted sum $Z$ that "splits" and is equivalent to the usual direct sum $X \oplus Y$. Any sufficiently non-linear but quasi-linear map $F$ produces a non-trivial twisted sum, the one that does not split. In particular, going back to the original three-space problem for the Hilbert spaces $X = Y = \ell_2$, Kalton and Peck choose

$$F\Big(\sum_i x_i e_i\Big) = \sum_i \big(\log \|x\| - \log |x_i|\big) x_i e_i.$$

This produces the *the Kalton-Peck's space* $Z = Z_2$, a twisted sum of Hilbert spaces which itself is not isomorphic to a Hilbert space.

Of course, such construction based on a *quasi*-linear map $F$ can only get us a *quasi*-Banach space $Z$ even when $X$ and $Y$ are Banach spaces. On the one hand, this means that perhaps quasi-Banach rather than Banach spaces are the most natural habitat for twisted sums. On the other hand, an inspection of an earlier construciton of Enflo, Lindenstrauss and Pisier [1] led Kalton [3] to a crucial realization that *a posteriori* $Z$ is often (but not always) isomorphic to a Banach space. Fortunately, this always happens when $X$ and $Y$ are Hilbert spaces, and more generally if $X$ and $Y$ are Banach spaces of *non-trivial type* (such are $\ell_p$ spaces with $1 < p < \infty$). So, most twisted sums automatically stay within the category of Banach spaces, even though their canonical construction goes deeply through quasi-Banach spaces!

Following its original discovery, Kalton-Peck's space $Z_2$ has become one of the main testing grounds for results and conjectures in Banach space theory. Its story continues up to the present day.

## References

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