# Isoperimetry of waists and local versus global asymptotic convex geometries

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**Abstract**

If two symmetric convex bodies $K$ and $L$ both have nicely bounded sections, then the intersection of random rotations of $K$ and $L$ is also nicely bounded. For $L$ being a subspace, this main result immediately yields the unexpected "existence vs. prevalence" phenomenon: *If $K$ has one nicely bounded section, then most sections of $K$ are nicely bounded.* The main result represents a new connection between the local asymptotic convex geometry (study of sections of convex bodies) and the global asymptotic convex geometry (study of convex bodies as a whole). Our method relies on the recent "isoperimetry of waists" due to M.Gromov.

## 1 Introduction

In convex geometry, there exists a remarkable parallelism between the local and the global asymptotic theories, as discovered by Milman in Schechtman in [MS] and further developed in [GM1], [GM2], [LMS]. Given a convex body $K$ symmetric about the origin in $\mathbb{R}^n$, the local structure of $K$ is given by its sections through the origin, and the global structure of $K$ is given by the interections of rotations of $K$. Some connections between local and global structures have been noticed long ago. Local and global geometries of the cube were handled simultaneously by Kashin [K] (see also [P]). The work [MS] developed a link between the local and the global structures of a general convex body $K$, which manifests itself when the dimension $n$ of the ambient space grows to infinity.

Within this "local versus global" paradigm, we show that a new phenomenon occurs, which could be described as *"existence versus prevalence"*. A deterministic

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local information (for *some* section) may imply a random global information (for *most* rotations) – and from there one may go back to a random local information. This shows that for certain natural problems in convex geometry, the Probabilistic Method must work if any other method works. Our main result is the following theorem:

**Theorem 1.1.** *There exists positive absolute constants $c$ and $C$ such that the following holds. Assume that two symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ have sections of dimensions at least $k$ and $n - ck$ respectively whose diameters are bounded by 1. Then for a random orthogonal operator $U \in \mathcal{O}(n)$, the body $K \cap UL$ has diameter bounded by $C^{n/k}$ with probability at least $1 - e^{-n}$.*

This result seems to be nontrivial already for $K = L$. The exponential bound $C^{n/k}$ can be improved to a polynomial bound, say $C(n/k)^2$, at the cost of decreasing the probability from $1 - e^{-n}$ to $1 - e^{-k}$. This is proved in the Appendix by M.Rudelson and the author.

The principal new idea in our approach to Theorem 1.1 is to make use of the recent *isoperimetry of waists* due to Gromov [G]. This produces a rather flexible method that can potentially be used in other problems. We outline this method in the end of this section.

An important direct consequence of Theorem 1.1 is that existence implies prevalence for boundedness of the diameters of sections of convex symmetric bodies $K$: *if $K$ has a nicely bounded section, then most sections of $K$ are nicely bounded* (with a certain loss of the diameter and of the dimension). This phenomenon was also independently discovered by A.Giannopoulos, V.D.Milman and A.Tsolomitis in their forthcoming work [GMT]. Precisely, for $L = \mathbb{R}^{ck}$ or $K = \mathbb{R}^{n-k}$, Theorem 1.1 immediately yields

**Corollary 1.2 (Propagation of boundedness of sections).** *There exists positive absolute constants $c$ and $C$ such that the following holds. Let $K$ be a convex symmetric body in $\mathbb{R}^n$ and $k$ be a positive integer.*

*(i) If there* **exists** *a section of $K$ of dimension $k$ whose diameter is bounded by 1, then a* **random** *section of $K$ of dimension $ck$ has diameter bounded by $C^{n/k}$ with probability at least $1 - e^{-cn}$.*

*(ii) If there* **exists** *a section of $K$ of dimension $n - ck$ whose diameter is bounded by 1, then a* **random** *section of $K$ of dimension $n - k$ has diameter bounded by $C^{n/k}$ with probability at least $1 - e^{-cn}$.*

The randomness here is with respect to the Haar measure on the Grassmanian $G_{n,m}$.

The forthcoming paper [GMT] offers better bounds on the diameter (note also that the version of Theorem 1.1 in the Appendix gives polynomial bounds and, moreover, probabilistic tail estimates on the diameter).

One previously known result in asymptotic convex geometry in which existence implied prevalemce was the Spreading Lemma due to Mankiewicz and Tomczak-Jaegermann ([MT] Proposition 3.2). It states that if a symmetric convex body $K$ lies in some subspace $E \subset \mathbb{R}^n$ of dimension $m = \lambda n$ and if most of the orthogonal projections of $K$ in $E$ of fixed rank $k < m$ contain a unit Euclidean ball, then most of the orthogonal projections of $K$ in $\mathbb{R}^n$ of dimension $k$ must also contain a Euclidean ball. The radius of that ball depends only on $\lambda$ (and deteriorates as $\lambda \to 1$). The Spreading Lemma has been successfully used in the geometric functional analysis. By duality, it implies a weaker form of Corollary 1.2, with the assumption that there exists a projection $PK$ (instead of a section) of $K$ such that most sections of $PK$ (of fixed dimension) have diameter bounded by 1.

As a typical application of our results, one can turn information about local structure of convex bodies into information about their global structure. Let us show this on the example of the volume ratio theorem, one of the important "local" results in the field.

**Corollary 1.3 (Global volume ratio theorem).** *There exists positive absolute constants $c$ and $C$ such that the following holds. Assume that a convex body $K$ in $\mathbb{R}^n$ contains the unit Euclidean ball $D$, and let $A = (|K|/|D|)^{1/n}$. Assume that a convex symmetric body $L$ in $\mathbb{R}^n$ has a section of dimension $k$ whose diameter is bounded by 1. Then for a random orthogonal operator $U \in \mathcal{O}(n)$ the body $K \cap UL$ has diameter bounded by $(2A)^{Cn/k}$ with probability at least $1 - e^{-n}$.*

For $L = \mathbb{R}^{n-k}$, Corollary 1.3 is the classical volume ratio theorem due to Szarek and Tomczak-Jaegermann ([S], [ST], see [P] Chapter 6); the best constant in this case is known to be $C = 1$ (with $4\pi$ replacing the factor of 2).

**Proof.** By Rogers-Shephard [RS], the volume of $K' = K - K$ is $|K'| \le \binom{2n}{n}|K| \le 4^n|K|$. Then $K'$ is symmetric, contains $D$ and $(|K'|/|D|)^{1/n} \le 4A$. By the volume ratio theorem (see e.g. [P]), $K'$ has a section of dimension at least $n - ck$ whose diameter is bounded by $M = (16\pi A)^{n/ck}$. The proof is finished by applying Theorem 1.1 to $M^{-1}K'$ and $L$. ■

Our approach to the proof of Theorem 1.1 is based on a recent isoperimetric theorem of M.Gromov [G], his "isoperimetry of waists" on the unit Euclidean sphere $S^{n-1}$:

> If $f : S^{k-1} \to S^{n-1}$ is an odd and continuous map, then the $(n-1)$-volume of any $\varepsilon$-neighborhood of $f(S^{k-1})$ in the geodesic distance on the sphere is minimized when $f$ is the canonical embedding, i.e. when the "waist" $f(S^{k-1})$ is an equatorial sphere.

This is proved in [G] for certain $k$ and $n$ and it remains an open problem for the rest of $k, n$; see next section.

The isoperimetry of waists can be effectively used in the asymptotic convex geometry. Suppose we know that for a symmetric convex body $K$ in $\mathbb{R}^n$ *there exists an orthogonal projection $PK$ that contains the unit Euclidean ball.* Without loss of generality, let $S^{k-1}$ be the sphere of that ball. One can find an odd and continuous lifting $g : S^{k-1} \to K$ of the projection $P$ and contract it to the sphere by defining $f(x) = g(x)/|g(x)|$. Then $f : S^{k-1} \to S^{n-1}$ satisfies the assumptions of Gromov's isoperimetry and, moreover, the waist $f(S^{k-1})$ lies in $K$. Then the isoperimetry of waists gives a *computable lower bound on the $(n-1)$-volume of any $\varepsilon$-neighborhood of $K$ on the sphere $S^{n-1}$.* This bound is sharp; it reduces to an equality if the projection $PK$ coincides with the section $K \cap P\mathbb{R}^n$. The exact statement is Proposition 3.1.

This argument is the main step in the proof of (the dual form of) Theorem 1.1. The assumptions are that both $K$ and $L$ have orthogonal projections that contain unit Euclidean balls. The reasoning above based on the isoperimetry of waists implies that the appropriate neighborhoods $K_{\varepsilon_1}$ of $K$ and $L_{\varepsilon_2}$ of $L$ have large $(n-1)$-volumes on the unit sphere. Then a standard $\varepsilon$-net argument shows that the Minkowski sum $K_{\varepsilon_1} + UL_{\varepsilon_2}$ contains the unit Euclidean ball with large probability (see Lemma 4.1). If $\varepsilon_1 + \varepsilon_2$ is a small number, then $K + UL$ must contain some nontrivial Euclidean ball, too. This is (the dual form of) the conclusion of Theorem 1.1.

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## 2 Gromov's isoperimetry of waists

Absolute constants will be denoted by $C$, $c$, $c_1$ ... Their values may be different in different occurences. The normalized Lebesgue measure on the unit Euclidean sphere $S^n$ will be denoted by $\sigma_n$. For a subset $A \subset S^n$ and a number $\theta > 0$, by $A_\theta$ we denote the $\theta$-neighborhood of $A$ in the geodesic distance $d$, i.e. $A_\theta = \{y \in S^n : d(x,y) \le \theta \text{ for some } x \in A\}$. A map $f$ is called odd is $f(-x) = -f(x)$ for all $x$. The following isoperimetry is proved in [G] 6.3.B.

**Theorem 2.1 (Gromov's isoperimetry of waists).** *Let $n$ be odd and $l = 2^k - 1$*

*for some integer $k$. Let $f : S^{n-l} \to S^n$ be an odd continuous function. Then for all $0 < \theta < \pi/2$*

$$\sigma_n((f(S^{n-l}))_\theta) \geq \sigma_n((S^{n-l})_\theta).$$

Conjecturally, this theorem should hold for all $n, l$. We will actually need this for all $n, l$, and in the absence of such result we will deduce a relaxed version of Gromov's theorem for all $n, l$, Corollary 2.3 below. This is done naturally by embedding into a higher dimensional sphere.

**Lemma 2.2.** *Let $A \subset S^n$ be an origin symmetric measurable set and $m \geq n$ be a positive integer. Then for all $0 < \theta < \pi/2$ one has $\sigma_n(A_\theta) \geq \sigma_m(A_\theta)$, where in the right side we look at a set $A$ as a subset of $S^m$ via the canonical embedding $S^n \subset S^m$.*

**Proof.** Fix an $x \in S^m$ and let $x_1$ be its spherical projection onto $S^n$, i.e. $x_1 = P_n x/|P_n x|$, where $P_n$ denotes the orthogonal projection in $\mathbb{R}^{m+1}$ onto $\mathbb{R}^{n+1}$.

CLAIM: $d(x_1, A) \leq d(x, A)$.

To prove the claim, since $A$ is symmetric it is enough to check that $d(x_1, a) \leq d(x, a)$ for all $a \in A$ such that $d(x, a) \leq \pi/2$. Since $0 \leq d(x, a) \leq \pi/2$ and $\langle x, a\rangle = \cos d(x, a)$, we have $0 \leq \langle x, a\rangle \leq 1$. Since $a \in S^n$, we have $P_n a = a$; thus

$$\langle x_1, a\rangle = \langle P_n x/|P_n x|, a\rangle = \frac{1}{|P_n x|}\langle x, a\rangle \geq \langle x, a\rangle.$$

In particular, $0 \leq \langle x_1, a\rangle \leq 1$. Since the function $\cos^{-1} : [0, 1] \to [0, \pi/2]$ is decreasing, we have $d(x_1, a) = \cos^{-1}\langle x_1, a\rangle \leq \langle x, a\rangle = d(x, a)$. This proves the Claim.

Now we can finish the proof of the lemma as follows:

$$\begin{aligned}\sigma_m(A_\theta) = \sigma_m(x \in S^m :\ d(x, A) \leq \theta) &\leq \sigma_m(x \in S^m :\ d(x_1, A) \leq \theta)\\ &= \sigma_n(x_1 \in S^n :\ d(x_1, A) \leq \theta) = \sigma_n(A_\theta)\end{aligned}$$

where the last line is obtained by representing a uniformly distributed vector $x \in S^m$ as $x = \gamma x_1 + \sqrt{1-\gamma}x_2$, where $x_1 \in S^n$ and $x_2 \in S^{m-n}$ are uniformly distributed, $\gamma$ is an appropriate random variable and the three random variables $x_1, x_2, \gamma$ are jointly independent. ■

**Corollary 2.3 (General (relaxed) isoperimetry of waists).** *Let $l < n$ be positive integers. Let $f : S^{n-l} \to S^n$ be an odd continuous function. Then for all $0 < \theta < \pi/2$*

$$\sigma_n((f(S^{n-l}))_\theta) \geq \sigma_{n+l+1}((S^{n-l-1})_\theta).$$

**Proof.** CASE 1: $n-l$ is even. Let $k$ be the minimal integer such that $2^k - 1 \geq l$. Then $m := (n-l) + (2^k - 1)$ is odd. Moreover, since $2^{k-1} < l$, we have $2^k \leq 2(l+1)$, so $m < n - l + 2(l+1) - 1 \leq n + l + 1$. Hence

$$n \leq m \leq n + l.$$

Then Gromov's theorem can be applied to functions from $S^{n-l} \to S^m$, in particular to $f : S^{n-l} \to S^n \to S^m$ where the second map is the canonical embedding. Then using Lemma 2.2, Gromov's theorem and Lemma 2.2 again, we have

$$\sigma_n((f(S^{n-l}))_\theta) \geq \sigma_m((f(S^{n-l}))_\theta) \geq \sigma_m((S^{n-l})_\theta) \geq \sigma_{n+l}((S^{n-l})_\theta). \tag{2.1}$$

Since by Lemma 2.2, $\sigma_{n+l}((S^{n-l})_\theta) \geq \sigma_{n+l+1}((S^{n-l})_\theta) \geq \sigma_{n+l}((S^{n-l-1})_\theta)$, the proof is complete in Case 1.

CASE 2: $n-l$ is odd. Apply Case 1 to the function $g : S^{n-l-1} \to S^{n-l} \to S^n$ where the first map is the canonical embedding and the second map is $f$. By (2.1), we have $\sigma_n((f(S^{n-l}))_\theta) \geq \sigma_n((g(S^{n-l-1}))_\theta) \geq \sigma_{n+l+1}((S^{n-l-1})_\theta)$. This completes the proof. ■

To simplify the use of Corollary 2.3, we will denote:

$$\sigma_{n,k}(\theta) = \sigma_n((S^k)_\theta), \quad \sigma_{n,k}^{\text{func}}(\theta) = \inf_f \sigma_n((f(S^k))_\theta),$$

where the infimum is over all symmetric continuous functions $f : S^k \to S^n$.

**Corollary 2.3′.** *Let $k < n$ be positive integers. Then for $0 < \theta < \pi/2$*

$$\sigma_{n,k}^{\text{func}}(\theta) \geq \sigma_{2n-k+1,k-1}(\theta). \tag{2.2}$$

**Remark.** If Gromov's theorem is true for all $n, l$, then Corollary 2.3′ improves to

$$\sigma_{n,k}^{\text{func}}(\theta) \geq \sigma_{n,k}(\theta).$$

The right hand side of (2.2) is a computable quantity. Sharp asymptotic estimates on $\sigma_{n,k}(\theta)$ were found by S.Artstein [A]. For our present purpose, we will be satisfied with much less precise estimates, which are elementary and well known.

**Lemma 2.4.** *Let $1 < k \leq n$ be integers and let $0 < \varepsilon < 1/2$. Then*

$$(c\varepsilon)^{2k} \leq \sigma_{n-1,n-k-1}(\sin^{-1}\sqrt{\frac{\varepsilon^2 k}{n}}) \leq (C\varepsilon)^{k/2}.$$

*Consequently,*

$$1 - (C\varepsilon)^{k/2} \leq \sigma_{n-1,k-1}(\sin^{-1}\sqrt{1 - \frac{\varepsilon^2 k}{n}}) \leq 1 - (c\varepsilon)^k.$$

When the general Gromov's theorem (Corollary 2.3$'$) is combined with Lemma 2.4, we obtain explicit estimates for $\sigma_{n,k}^{\text{func}}(\theta)$:

**Corollary 2.5.** *Let* $1 < k \leq n$ *be integers and let* $0 < \varepsilon < 1/2$. *Then*

*(i)* $\sigma_{n-1,n-k-1}^{\text{func}}(\sin^{-1}\sqrt{\frac{\varepsilon^2 k}{n}}) \geq (c\varepsilon)^{8k}$;

*(ii)* $\sigma_{n-1,k-1}^{\text{func}}(\sin^{-1}\sqrt{1-\frac{\varepsilon^2 k}{n}}) \geq 1-(C\varepsilon)^{k/4}$.

**Proof.** Let $\alpha = k/n$.

(i) By Corollary 2.3$'$,

$$\sigma_{n-1,(1-\alpha)n-1}^{\text{func}}(\sin^{-1}\sqrt{\varepsilon^2\alpha}) \geq \sigma_{n+\alpha n,(1-\alpha)n-2}(\sin^{-1}\sqrt{\varepsilon^2\alpha}). \tag{2.3}$$

To apply Lemma 2.4, write the right hand side of (2.3) for suitable $m$ and $\beta$ as

$$\sigma_{m-1,(1-\beta)m-1}(\sin^{-1}\sqrt{(\varepsilon^2\alpha/\beta)\cdot\beta}) \geq (c\sqrt{\varepsilon^2\alpha/\beta})^{2\beta m}. \tag{2.4}$$

The numbers $m$ and $\beta$ are, of course, determined by $m-1 = n+\alpha n$ and $(1-\beta)m-1 = (1-\alpha)n-2$. Hence $\beta = (2\alpha n+2)/(n+\alpha n+1)$, so that $\alpha < \beta < 3\alpha$. Then we can continue (2.4) as

$$\geq (c_1\varepsilon)^{4(\alpha n+1)} \geq (c_1\varepsilon)^{8k}.$$

This completes part (i).

(ii) By Corollary 2.3$'$,

$$\sigma_{n-1,\alpha n-1}^{\text{func}}(\sin^{-1}\sqrt{1-\varepsilon^2\alpha}) \geq \sigma_{2n-\alpha n,\alpha n-2}(\sin^{-1}\sqrt{1-\varepsilon^2\alpha}). \tag{2.5}$$

To apply Lemma 2.4, write the right hand side of (2.5) for suitable $m$ and $\beta$ as

$$\sigma_{m-1,\beta m-1}(\sin^{-1}\sqrt{1-(\varepsilon^2\alpha/\beta)\cdot\beta}) \geq 1-(C\sqrt{\varepsilon^2\alpha/\beta})^{\beta m} - e^{-10m}. \tag{2.6}$$

The numbers $m$ and $\beta$ are, of course, determined by $m-1 = 2n-\alpha n$ and $\beta m-1 = \alpha n-2$. Hence $\beta = (\alpha n-1)/(2n-\alpha n+1)$, so that $\beta \geq \alpha/2$. Then we can continue (2.6) as

$$\geq 1-(C_1\varepsilon)^{(\alpha n-1)/2} \geq 1-(C_1\varepsilon)^{k/4}.$$

This completes part (ii). ■

## 3 Waists generated by projections of convex bodies

The following observation connects the isoperimetry of waists to convex geometry.

For simplicity, given a set $A \in \mathbb{R}^n$ we write $\sigma_{n-1}(A)$ for $\sigma_{n-1}(A \cap S^{n-1})$, if measurable. The unit Euclidean ball in $\mathbb{R}^n$ is denoted by $D$. Minkowski sum in $\mathbb{R}^n$ is defined as $A+B = \{a+b : \ a \in A, \ b \in B\}$.

**Proposition 3.1.** *Let $K$ be a convex symmetric set in $\mathbb{R}^n$. Assume there is an orthogonal projection $P$,* $\mathrm{rank} P = k$*, such that $PK \supseteq PD$. Then for all* $0 < \varepsilon < 1$

$$\sigma_{n-1}(K + \varepsilon D) \geq \sigma^{\mathrm{func}}_{n-1,k-1}(\sin^{-1} \varepsilon). \tag{3.1}$$

**Remark.** The power of this fact is that the right side of (3.1) is easily estimated via Gromov's theorem (Corollary 2.5).

**Proof.** We can assume that the range of $P$ is $\mathbb{R}^k$, so $PK \supseteq S^{k-1}$. According to Bartle-Graves selection theorem [BG] (which Michael's selection theorem [Mich] generalizes), there exists a continuous selection $g : S^{k-1} \to K$ of the projection $P$, i.e. $Pg$ is the identity. There is no problem in assuming that $g$ is an odd function by redefining $g(x)$ to be $\frac{1}{2}(g(x) - g(-x))$.

$$f : S^{k-1} \to S^{n-1}, \quad f(x) = g(x)/|g(x)|.$$

The function $f$ is odd and it is continuous because $|g(x)| \geq |Pg(x)| = |x| = 1$ for all $x$. So $f(x)$ is a contraction of the point $g(x)$ onto the sphere. Since $g(x) \in K$ and $K$ is star-shaped, we have $f(x) \in K$, thus

$$f(S^{k-1}) \subseteq K \cap S^{n-1}.$$

By making a simple planar drawing, one sees that for every $y \in S^{n-1}$

$$[-y, y] + \varepsilon D \supseteq \{y\}_{\sin^{-1} \varepsilon}. \tag{3.2}$$

Running $y$ over $f(S^{k-1})$, we obtain

$$\begin{aligned} K + \varepsilon D &\supseteq f(S^{k-1}) + \varepsilon D \\ &= \bigcup_{y \in f(S^{k-1})} \Big([-y, y] + \varepsilon D\Big) \quad \text{by the symmetry of } f \\ &= f(S^{k-1})_{\sin^{-1} \varepsilon} \quad \text{by (3.2).} \end{aligned}$$

Intersecting both sides with $S^{n-1}$ and taking the measure completes the proof. ■

Proposition 3.1 will in particular be used to estimate the covering number of $K + \varepsilon D$. Given two convex sets $L$ and $K$, the covering number $N(L, K)$ is the minimal number of translates of $K$ needed to cover $L$. By a simple and known volumetric argument, $N(L, K) \leq \frac{|L+K|}{|K|}$.

**Lemma 3.2.** *For every convex symmetric set $K$,*

$$N(D, K) \leq 2^n / \sigma_{n-1}(K).$$

**Proof.**
$$N(D,K) \leq N(D, K\cap D) \leq \frac{|D+(K\cap D)|}{|K\cap D|} \leq \frac{|2D|}{|K\cap D|}. \tag{3.3}$$

Next,
$$\sigma_{n-1}(K) = \sigma_{n-1}(K\cap D) \leq \frac{|K\cap D|}{|D|}, \tag{3.4}$$

which folows from a standard argument that transfers the surface measure on $S^{n-1}$ to the volume in $D$ (a set $A \subseteq S^{n-1}$ generates the cone $\cup_{0<t<1} tA$, which occupies the same portion of the volume in $D$ as $\sigma_{n-1}(A)$). Then (3.3) and (3.4) complete the proof. ■

# 4 Proof of Theorem 1.1

By duality, Theorem 1.1 can equivalently be stated as follows. There exist an absolute constant $a \in (0,1)$ such that the following holds. Assume that there exist orthogonal projections $P$ and $Q$ with $\mathrm{rank} P = k$ and $\mathrm{rank} Q = n - ak$, and such that
$$PK \supseteq PD, \quad QL \supseteq QD. \tag{4.1}$$

Then for $U$ as in the theorem, we claim that
$$K + UL \supseteq C^{n/k} D. \tag{4.2}$$

The idea is as follows. Let $\delta_K, \delta_L > 0$ be parameters. By Gromov's theorem and Lemma 3.2, we will be able to estimate
$$1-\sigma := \sigma_{n-1}(K+\delta_K D) \quad \text{and} \quad N := N(2D, L+\delta_L D). \tag{4.3}$$

**Lemma 4.1.** *Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ such that* (4.3) *holds and $\delta_K + \delta_L < 1$. Then for a random orthogonal operator $U \in \mathcal{O}(n)$*
$$(1-\delta_K-\delta_L)D \subseteq K + UL \tag{4.4}$$
*with probability at least $1 - N\sigma$.*

**Proof.** By a standard argument, the sphere $S^{n-1}$ of $D$ can be covered by $N$ translates of the body $L + \delta_L D$ by vectors from $S^{n-1}$. Hence there exists a subset $\mathcal{N} \subset S^{n-1}$ such that
$$|\mathcal{N}| = N \quad \text{and} \quad D \subseteq \mathcal{N} + L + \delta_L D. \tag{4.5}$$

Since for every $z \in S^{n-1}$, its image $Uz$ under a random rotation $U \in \mathcal{O}(n)$ is uniformly distributed on the sphere, we have for any fixed $z \in \mathcal{N}$:

$$\mathbb{P}\{U \in \mathcal{O}(n) \,:\, Uz \in K + \delta_K D\} = \sigma_{n-1}(K + \delta_K D) = 1 - \sigma.$$

Thus

$$\mathbb{P}\{U \in \mathcal{O}(n) \,:\, U\mathcal{N} \subseteq K + \delta_K D\} \geq 1 - N\sigma.$$

Fix any $U$ in this set and apply it to the inclusion in (4.5):

$$D \subseteq U\mathcal{N} + UL + \delta_L D \subseteq K + \delta_K D + UL + \delta_L D.$$

Since $\delta_K + \delta_L < 1$, this inclusion implies (4.4). ■

**Proof of Theorem 1.1.** We can clearly assume that $0 < a < 1/33$ and that $ak \geq 1$. Let

$$\varepsilon_K > 0, \quad \delta_K = \sqrt{1 - \frac{\varepsilon_K^2 k}{n}}.$$

By Proposition 3.1 and Corollary 2.5 (ii),

$$\sigma_{n-1}(K + \delta_K D) \geq \sigma_{n-1,k-1}^{\text{func}}(\sin^{-1}\delta_K) \geq 1 - (C\varepsilon_K)^{k/4} \geq 1 - 2e^{-10n}$$

if one chooses the value of $\varepsilon_K$ as

$$\varepsilon_K = \exp(-C_1 n/k),$$

where $C_1 > 0$ is a sufficiently large absolute constant. Similarly, let

$$\varepsilon_L > 0, \quad \delta_L = \sqrt{\frac{\varepsilon_L^2 ak}{n}}.$$

By Proposition 3.1 and Corollary 2.5 (i),

$$\sigma_{n-1}(L + \delta_L D) \geq \sigma_{n-1,n-ak-1}^{\text{func}}(\sin^{-1}\delta_L) \geq (c\varepsilon_L)^{8k} \geq \frac{1}{2}e^{-n/2}$$

if one chooses the value of $\varepsilon_L$ as

$$\varepsilon_L = \exp(-c_2 n/ak)$$

where $c_2 > 0$ is a sufficiently small absolute constant. By Lemma 3.2,

$$N(2D, 2L + 2\delta_L D) = N(D, L + \delta_L D) \leq 2^n / \frac{1}{2}e^{-n/2} \leq 2e^{1.2n}.$$

By Lemma 4.1, if $\delta_K + 2\delta_L < 1$ then the desired inclusion

$$(1 - \delta_K - 2\delta_L)D \subseteq K + 2UL \tag{4.6}$$

holds with probability at least

$$1 - 2e^{1.2n} \cdot 2e^{-10n} \geq 1 - e^{-n}.$$

So it only remains to bound below

$$\delta_K + 2\delta_L = \sqrt{1 - \exp(-2C_1 n/k)(k/n)} + 2\sqrt{\exp(-2c_2 n/ak)(ak/n)}.$$

This can be quickly done using the inequalities $\sqrt{1-x} \leq 1 - x/2$ and $xe^{-C/x} \geq e^{-2C/x}$ valid for all $0 < x < 1$ and for a sufficiently large absolute constant $C$. We thus have

$$\delta_K + 2\delta_L \leq 1 - \frac{1}{2}\exp(-C_1' n/k) + 2\exp(-c_2' n/ak) < 1 - \frac{1}{4}\exp(-Cn/k)$$

if $a$ is chosen a sufficiently small absolute constant. This together with (4.6) completes the proof. ■

## 5 Appendix by Mark Rudelson and Roman Vershynin

In this appendix we show how to improve the exponential bound $C^{n/k}$ on the diameter of the random intersection $K \cap UL$ in Theorem 1.1 by a polynomial bound, say $C(n/k)^2$. The cost for this is decreasing the probability from $1 - e^{-n}$ to $1 - e^{-k}$. More generally, one can estimate the distribution of the diameter of the random intersection as follows.

**Theorem 5.1.** *Let $0 < a < 1/64$. Assume that two symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ have sections of dimensions at least $k$ and $n - ak$ respectively whose diameters are bounded by 1. Then for every $t > C(n/k)^{Ca}$ the random orthogonal operator $U \in \mathcal{O}(n)$ satisfies*

$$\mathbb{P}\Big\{\operatorname{diam}(K \cap UL) > tn/k\Big\} < (ct)^{-k/16}.$$

**Remarks.** **1.** Theorem 1.1 is a partial case of this theorem for $t = C_1^{n/k}$:

$$\mathbb{P}\Big\{\operatorname{diam}(K \cap UL) > C^{n/k}\Big\} < e^{-n}.$$

**2.** To obtain a polynomial bound on the diameter, one can choose $t = C_1(n/k)$ in Theorem 5.1 to get

$$\mathbb{P}\Big\{\operatorname{diam}(K \cap UL) > C_1(n/k)^2\Big\} < (cC_1 n/k)^{-k/16} < e^{-k}$$

for an appropriate absolute constant $C_1$. To summarize, *In Theorem 1.1, the body* $K \cap UL$ *has diameter bounded by* $C_1(n/k)^2$ *with probability at least* $1 - e^{-k}$.

A new ingredient in the proof of Theorem 5.1 is the following covering lemma.

**Lemma 5.2.** *Let* $K$ *be a convex body in* $\mathbb{R}^n$ *such that* $K \supseteq \delta D$ *for some* $\delta > 0$. *Assume that there exists an orthogonal projection* $P$ *with* $\mathrm{rank}P = n - k$ *and such that* $PK \supseteq PD$. *Then*

$$N(D, 4K) \le (C/\delta)^{2k}.$$

**Proof.** Denote the range of $P$ by $E$. Let $f : PD \to \mathbb{R}^n$ be a lifting of $f$, i.e. a map such that

$$f(PD) \subset K \quad \text{and} \quad V := (id - f)(PD) \subset E^\perp. \tag{5.1}$$

Since $V \subset PD - f(PD) \subset D - f(PD)$, the assumptions on $K$ and (5.1) imply that $V \subset (\frac{1}{\delta} + 1)K \cap E^\perp$. Then by the standard volumetric argument we have

$$N(V, K) \le (C/\delta)^k$$

as $\dim E^\perp = k$. This will allow us to cover $PD$. Indeed, by (5.1),

$$PD \subset f(PD) + V \subset K + V.$$

By the submultiplicative property $N(K_1 + K_2, D_1 + D_2) \le N(K_1, D_1)\ N(K_2, D_2)$, which is valid for all sets $K_1, K_2, D_1, D_2$, we have

$$N(PD, 2K) \le N(K + V, 2K) \le N(V, K) \le (C/\delta)^k. \tag{5.2}$$

Also by the assumption on $K$ and by the standard volumetric argument already used above,

$$N(D \cap E^\perp, K) \le N(D \cap E^\perp, \delta D \cap E^\perp) \le (C/\delta)^k. \tag{5.3}$$

Since $D \subset PD + D \cap E^\perp$, we have by the submultiplicative property that

$$N(D, 3K) \le N(PD, 2K)\ N(P \cap E^\perp, K)$$

and we finish by applying (5.2) and (5.3). ■

**Proof of Theorem 5.1.** We start the proof as in Section 4 by dualizing the statement and assuming that there exist orthogonal projections $P$ and $Q$ with $\mathrm{rank}P = k$ and $\mathrm{rank}Q = n - ak$, and such that (4.1) holds. Then we must prove that for $t$ as in the theorem,

$$\mathbb{P}\{(k/tn)D \subseteq K + UL\} \ge 1 - (ct)^{-k/16}. \tag{5.4}$$

Let $\varepsilon > 0$ and let

$$\delta_K = \sqrt{1 - \frac{\varepsilon^2 k}{n}}.$$

By Proposition 3.1 and Corollary 2.5 (ii),

$$\sigma_{n-1}(K + \delta_K D) \geq \sigma^{\text{func}}_{n-1,k-1}(\sin^{-1} \delta_K) \geq 1 - (C\varepsilon_K)^{k/4}.$$

Let $0 < \delta_L < 1$ be a parameter. By Lemma 5.2 applied to the body $L + \delta_L D$,

$$N(D, 4(L + \delta_L D)) \leq (C/\delta_L)^{2ak}.$$

Writing this covering number as $N(2D, 8L + 8\delta_L D)$, we apply Lemma 4.1. It states that if $\delta_K + 8\delta_L < 1$ then the inclusion

$$(1 - \delta_K - 8\delta_L)D \subseteq K + 8UL \tag{5.5}$$

holds with probability at least

$$1 - (C/\delta_L)^{2ak}(C\varepsilon)^{k/4}. \tag{5.6}$$

To finish the proof, we need to bound below the radius $1 - \delta_K - 8\delta_L$ in (5.5) and the probability (5.6). Since $\sqrt{1-x} \leq 1 - x/2$ for $0 < x < 1$, we set

$$\delta_L = \frac{\varepsilon^2 k}{32n}$$

to obtain

$$1 - \delta_K - 8\delta_L \geq \frac{\varepsilon^2 k}{4n}.$$

It remains to estimate the probability (5.6). If we require that for suitable absolute constants $c_0, C_0 > 0$

$$\varepsilon \leq c_0(k/n)^{C_0 a} \leq c_0(k/n)^{16a/(1-32a)} \tag{5.7}$$

(such a $C_0$ exists if, say, $0 < a < 1/64$), then $(C/\delta_L)^{2ak} < (C\varepsilon)^{-k/8}$, hence the probability

$$(5.6) \geq 1 - (C\varepsilon)^{k/8}.$$

We have thus proved that if $\varepsilon > 0$ satisfies (5.7) then

$$\mathbb{P}\{(\varepsilon^2 k/32n)D \subseteq K + UL\} \geq \mathbb{P}\{(\varepsilon^2 k/4n)D \subseteq K + 8UL\} \geq 1 - (C\varepsilon)^{k/8}.$$

It remains to set $\varepsilon^2/32 = 1/t$, and (5.4) is proved. ■

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