

Principle of Mathematical Induction

1. Prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.
2. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.
3. Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$.
Hint: to verify induction step, work with the right hand side with $n+1$ instead of n . Use the formula for $(a+b)^2$.
4. Prove that $n^2 > n+1$ for all integers $n \geq 2$.
Hint: use a more general statement of the Principle of Mathematical Induction where the base can be arbitrary instead of 1. This is Statement 1.2.3 in the book.
5. Decide for which integers the inequality $2^n > n^2$ is true, and prove your claim by induction.

Countable and uncountable sets

6. Prove that the union of two countable sets is countable. Formally, let A and B be countable subsets of some set S . Prove that $A \cup B$ is countable.
7. (a) Prove that the set of all two-element subsets of \mathbb{N} is countable.
Hint: Use the zig-zag counting from the proof that \mathbb{Q} is countable.
(b) For any $n \in \mathbb{N}$, prove that the set of all n -element subsets of \mathbb{N} is countable.
Hint: Proceed by induction on n .

8. (a) A real number x is called an *algebraic number* if it is a real root of some polynomial with integer coefficients, that is if

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

for some $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{N}$. (For example, all rational numbers, as well as $\sqrt{2}$ and $5^{-1/3}$ are algebraic numbers.) Prove that the set of algebraic numbers is countable.

Hint: use the Fundamental Theorem of Algebra, which states that such polynomial has at most n real roots.

9. (This logic problem will somehow pave the way for our discussion of continuity and limit.) Find the negations of the following statements.

- (a) Anyone living in Los Angeles has blue eyes.
- (b) Anyone living in Los Angeles who has blue eyes will win a lottery.
- (c) Anyone living in Los Angeles who has blue eyes will win a lottery and will take their retirement before the age of 50.

Supremum and infimum

10. Let S be a subset of \mathbb{R} which is bounded below. Prove that $\inf(S)$ exists, and that

$$\sup(-S) = -\inf(S).$$

(Here $-S$ denotes the set $\{-s : s \in S\}$.)

11. Let A and B be subsets of \mathbb{R} which are bounded above. Show that $A \cup B$ is bounded above, and

$$\sup(A \cup B) = \max(\sup A, \sup B).$$

12. Formulate and prove a statement analogous to Problem 11 about sets that are bounded *below*, and the infimum.

13. Let A and B be nonempty bounded subsets of \mathbb{R} . Prove the following:

- (a) $\sup(A + B) = \sup A + \sup B$;
- (b) $\inf(A + B) = \inf A + \inf B$.

(Here $A + B$ denotes the set $\{a + b : a \in A, b \in B\}$.)

14. Consider the set $S = \{1 - n^{-2} : n \in \mathbb{N}\}$. Show that $\sup S = 1$.

15. Consider the set $S = \{2^{-n} - 2^{-m} : n, m \in \mathbb{N}\}$. Find $\sup S$ and $\inf S$. (First guess these values, and then prove your claim.)

16. Let S be a bounded subset of \mathbb{R} . Show that

$$\sup(aS) = a \cdot \sup S; \quad \inf(aS) = a \cdot \inf S \quad \text{for every } a \geq 0.$$

(Here aS denotes the set $\{as : s \in S\}$.)

Do these identities hold for $a < 0$?

17. For the following sets, find the supremum and infimum, if they exist. (First guess these values, and then prove your claim.)

- (a) $S := \{x \in \mathbb{R} : x > 1/x\}$;
- (b) $T = \{n/(n+1) : n \in \mathbb{N}\}$.

18. Show that for each $a \in \mathbb{R}$.

$$\inf\{x \in \mathbb{Q} : x > a\} = a.$$

19. Prove that

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$

The definition of the limit

20. Compute the limits of the following sequences. (First guess the value and then prove your claim using the definition of the limit.)

(a) $\lim \frac{1}{\ln n}$.

(b) $\lim \frac{n^2 - 1}{2n^2 + 1}$.

(c) $\lim \frac{\sqrt{n}}{n + 3}$.

(d) $\lim \left(\frac{1}{n} - \frac{1}{n+2}\right)$.

- 21.** Consider a sequence (x_n) and let $y_n = |x_n|$.
(a) Prove that if (x_n) converges, then (y_n) converges, too.
(b) Show that the converse is false. (Give an example of a divergent sequence whose absolute values converge. No proof is needed, just an example.)

- 22.** Using the definition of the limit, prove that

$$\lim \frac{3^n}{n!} = 0.$$

Hint: try to bound the quantity $3^n/n!$ by some simpler quantity; perhaps by C/n with a suitable constant C .

- 23.** Let (x_n) be a convergent sequence, and $a \in \mathbb{R}$. Suppose that $x_n \geq a$ for all but finitely many $n \in \mathbb{N}$. Use the definition of the limit to show that

$$\lim x_n \geq a.$$

- 24.** According to the definition given in class, a sequence (x_n) is *bounded* if the set of its values $\{x_1, \dots, x_n\}$ is a bounded set in \mathbb{R} .

Show that (x_n) is bounded if and only if there exists $M \in \mathbb{R}$ such that

$$|x_n| \leq M \quad \text{for every } n \in \mathbb{N}.$$

- 25.** Prove that any convergent sequence satisfies

$$\inf\{x_n\} \leq \lim x_n \leq \sup\{x_n\}.$$

- 26.** Consider two sequences, (x_n) and (y_n) . Assume that for any $\varepsilon > 0$ there exists N such that

$$|x_n - y_n| < \varepsilon \quad \text{for all } n \geq N.$$

Prove that (x_n) converges if and only if (y_n) converges.

Limit theorems

From now on, feel free to use limit theorems.

27. Find the following limits.

(a) $\lim \frac{4\sqrt{n} + 3}{5\sqrt{n} + 2n^{0.1}}.$

(b) $\lim \frac{2n - 7}{5n^2 + n + 1}.$

(c) $\lim \sqrt{9n^2 + n} - 3n - 1.$

(d) $\lim n^{1/n^3}.$

28. Use limit theorems to compute limits in Problem 20 b, c, d.

29. Let $x_1 := 10$ and $x_{n+1} := \frac{1}{2}x_n + 2$. Show that the sequence (x_n) converges, and compute its limit.

30. Compute the value of the following continued fraction:

$$2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{\dots}}}}$$

Hint: interpret this value as a limit of a certain recursively defined sequence. Show that that sequence converges and compute its limit.

31. Let A be a subset of \mathbb{R} that is bounded below. Show that there exists a decreasing sequence (x_n) whose terms x_n are elements of A , and such that

$$\lim x_n = \inf A.$$

32. Prove that a sequence (x_n) is unbounded if and only if there exists a subsequence (x_{n_k}) that satisfies

$$\lim \frac{1}{x_{n_k}} = 0.$$

33. Find $\limsup x_n$ and $\liminf x_n$ for the sequence (x_n) defined as

$$x_n = (-1)^n + \frac{1}{n}.$$

Prove your claims.

34. Construct a sequence (x_n) so that for every real number x , there is a subsequence (x_{n_k}) that converges to x .

35. Let a, b be real numbers such that $|a| \neq |b|$. Determine when the sequence

$$x_n := \frac{a^n - b^n}{a^n + b^n}$$

converges and diverges, depending on the values of a and b .

36. Consider sequences (x_n) and (y_n) .

(a) Is it always true that

$$\limsup(x_n + y_n) = \limsup x_n + \limsup y_n?$$

Prove this or give a counterexample.

(b) Is it always true that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n?$$

Prove this or give a counterexample.

37. Let $x_1 := 1$, $x_2 := 2$ and $x_{n+1} := \frac{1}{4}x_n + \frac{3}{4}x_{n-1}$ for $n > 2$. Does the sequence (x_n) converge or diverge? Prove your claim.

38. Prove that if $\lim x_n = +\infty$ and a sequence (y_n) converges, then $\lim(x_n + y_n) = +\infty$.

39. Prove that (x_n) is bounded if and only if $\limsup |x_n|$ exists (as a finite number).

40. Calculate $\lim(n!)^{1/n}$. (Determine if the sequence converges, diverges to $+\infty$, or diverges but not to $+\infty$. Prove your claims.)

41. Let (x_n) be a monotone sequence. Prove that if there exists a convergent subsequence (x_{n_k}) then the entire sequence (x_n) converges.

Series

42. Suppose that $\sum x_n$ is a series with positive terms. Show that if $\sum x_n$ converges then $\sum \frac{1}{x_n}$ diverges. What about the converse to this statement?

43. Prove the following facts:

- (a) If $\sum x_n$ and $\sum y_n$ both converge then $\sum (x_n + y_n)$ converges.
- (b) Let $n_0 \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} x_n$ converges if and only if the series $\sum_{n=n_0}^{\infty} x_n$ converges.

44. Prove the following powerful *Cauchy condensation test* for series $\sum x_n$ such that $x_1 \geq x_2 \geq \cdots \geq 0$. The series $\sum x_n$ converges if and only if the series

$$x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots = \sum 2^k x_{2^k}$$

converges.

Hint: Group the terms like we did when we analyzed harmonic series, and bound each group above and below. Specifically, argue that $x_2 \leq x_2 \leq x_1$, $2x_4 \leq x_3 + x_4 \leq 2x_2$, $4x_8 \leq x_5 + x_6 + x_7 + x_8 \leq 4x_4$, etc. Summing these inequalities will help you compare the partial sums of the two series in question.

45. Using the Cauchy condensation test from the previous problem, determine if the following series converge or diverge.

(a) $\sum \frac{1}{n}$ (harmonic series)

(b) $\sum \frac{1}{n \ln n}$

(c) $\sum \frac{1}{n \ln^2 n}$

46. Let $a_n \geq 0$, $b_n \geq 0$ for all n . Show that if $\sum a_n^2$ and $\sum b_n^2$ converge then $\sum a_n b_n$ converges.

Hint: think about the Comparison Test.

47. Determine if the following series converge or diverge.

(a) $\sum \frac{2n}{n^3 + 1}$

(b) $\sum \frac{\cos^2 n}{n^2}$

(c) $\sum \frac{1}{\sqrt{n!}}$

48. Compute the following limits if they exist. Prove your claims.

(a) $\lim_{x \rightarrow 0} \sqrt{x}$

(b) $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$

(c) $\lim_{x \rightarrow 0} \frac{x}{|x|}$

(d) $\lim_{x \rightarrow 0} \frac{x^{3/2}}{|x|}$

(e) $\lim_{x \rightarrow 0} \cos(1/x^2)$

(f) $\lim_{x \rightarrow 0} \sqrt{x} \cos(1/x^2)$

(g) $\lim_{x \rightarrow 1} \frac{\sqrt{1 + 3x^2} - 2}{x - 1}$

49. Define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by letting $f(x) := x$ for rational x and $f(x) := 2$ for irrational x . Find all points c at which f has a limit.

50. Compute the following limits if they exist. Prove your claims.

(a) $\lim_{x \rightarrow +\infty} e^{-x^2}$

(b) $\lim_{x \rightarrow 0+} e^{-1/x^3}$ and $\lim_{x \rightarrow 0-} e^{-1/x^3}$

(c) $\lim_{x \rightarrow +\infty} p(x)$ and $\lim_{x \rightarrow -\infty} p(x)$ for a general polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$.

$$(d) \lim_{x \rightarrow +\infty} \frac{2\sqrt{x} - x}{2\sqrt{x} + x}$$

$$(e) \lim_{x \rightarrow 0} \frac{\sqrt{1 + 3x^2} - 1}{x^2}$$

51. Formulate and prove a result that relates to each other the following two identities: $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} 1/f(x) = +\infty$.

52. Formulate and prove a result that relates to each other the following two limits: $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow 0+} f(1/x)$.

53. Let $a < b < c$ and consider two continuous functions $f : [a, b] \rightarrow \mathbb{R}$ and $g : [b, c] \rightarrow \mathbb{R}$.¹ Now glue them together by defining the function

$$h(x) := \begin{cases} f(x) & \text{if } x \in [a, b) \\ g(x) & \text{if } x \in [b, c]. \end{cases}$$

Find conditions under which h is continuous. Prove your claim.

54. Prove that the function $f(x) = |x|$ is continuous on \mathbb{R} .

55. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *additive* if

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that if a continuous function f is additive, then there exists a constant b such that

$$f(x) = bx \quad \text{for all } x \in \mathbb{R}.$$

Hint: first show the conclusion for $x \in \mathbb{N}$, then for $x \in \mathbb{Q}$, then for $x \in \mathbb{R}$. This should work with $b := f(1)$.

¹When we say “continuous function” we mean that it is continuous on its entire domain.

56. Let $f, g : D \rightarrow \mathbb{R}$ be continuous functions. Prove that

$$h(x) := \max(f(x), g(x))$$

is a continuous function.

Hint: Prove and then use the identity

$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2}, \quad a, b \in \mathbb{R}.$$

57. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function that takes strictly positive values, i.e. $f(x) > 0$ for all $x \in [a, b]$. Show that there exists $\delta > 0$ such that

$$f(x) > \delta \quad \text{for all } x \in [a, b].$$

58. Consider a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$. Show that if the degree n is odd, then $p(x)$ has a real root (i.e. there exists $x \in \mathbb{R}$ such that $p(x) = 0$.)

59. In this problem, you will mathematically demonstrate the following fact: there are, at any time, antipodal points on the earth's equator that have the same temperature.

Let $T(x)$ denote the temperature at the point of the equator with longitude x , where x ranges from 0° to 360° . We assume that T is a continuous function, and that $T(0) = T(360)$ since 0° and 360° denote the same point on the equator. The desired conclusion is that there exists $c \in [0, 180]$ such that

$$T(c) = T(c + 180).$$

Prove this conclusion.

Hint: consider the function $f(x) = T(x) - T(x + 180)$ defined on $[0, 180]$.

60. (a) Prove that the function $f(x) = 1/x$ is uniformly continuous on $[a, \infty]$ for every $a > 0$, but is not uniformly continuous on $(0, \infty)$.

(b) Prove that the function $f(x) = \sin(1/x)$ is not uniformly continuous on $(0, 1)$.

(c) Prove that the function

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous on $[-a, a]$, where $a > 0$ is a fixed real number.

(d) Prove that the function $f(x) = x^2$ is uniformly continuous on $[-R, R]$ for every $R > 0$, but is not uniformly continuous on \mathbb{R} .

61. Prove that a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is injective if and only if f is strictly monotone on $[a, b]$.

62. Compute $f'(0)$ or show it does not exist for the following functions. Prove your claims. For each of the examples, we set $f(0) = 0$, and for $x \neq 0$ we define f as follows:

(a) $f(x) = \sin(1/x)$

(b) $f(x) = x^2 \sin(1/x)$

(c) $f(x) = x \sin(1/x)$

(d) $f(x) = \sqrt{x}$

63. Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable at a single point and not differentiable anywhere else.

Hint: define the function differently for rational and irrational points.

64. (Landau's calculus) Recall that for two functions f and g , we say that

$$f(z) = o(g(z)) \text{ as } z \rightarrow 0 \quad \text{if} \quad \lim_{z \rightarrow 0} \frac{f(z)}{g(z)} = 0$$

and $f(z) = O(g(z))$ as $z \rightarrow 0$ if the function f/g is bounded on some neighborhood of zero, that is

$$\exists \delta > 0, M \geq 0 \text{ such that } |f(z)| \leq M|g(z)| \text{ for all } z \in (-\delta, \delta).$$

Interpret and prove each of the following statements formally, as $z \rightarrow 0$. (In other words, make it into a rigorous limit theorem using functions f , g , etc.)

(a) $o(z) + o(z) = o(z)$.

Hint: a formal statement here is the following. For any functions f and h satisfying $\lim_{z \rightarrow 0} f(z)/z = 0$ and $\lim_{z \rightarrow 0} h(z)/z = 0$, one has $\lim_{z \rightarrow 0} (f(z) + h(z))/z = 0$.

(b) $O(1) \cdot O(z) = O(z)$.

(c) $O(z) + o(z) = O(z)$.

(d) $o(O(z)) = o(z)$.

65. (Change of variables in the limit) Let f and g be functions defined on appropriate domains. (For simplicity, assume they are defined on all of \mathbb{R} .)

(a) Show that if

$$\lim_{x \rightarrow u} f(x) = v \quad \text{and} \quad \lim_{y \rightarrow v} g(y) = w$$

then

$$\lim_{x \rightarrow u} g(f(x)) = w.$$

Hint: this should be similar to the proof of the theorem about composition of continuous functions.

(b) Show that

$$\lim_{x \rightarrow 1} \cos \left(\frac{\sqrt{x} - 1}{x - 1} \right).$$

66. Prove the following inequality for all $x, y \in \mathbb{R}$:

$$|\cos x - \cos y| \leq |x - y|.$$

Hint: use Mean Value Theorem.

67. Let $f : I \rightarrow \mathbb{R}$ be a function that is differentiable on an interval I . Prove that f is Lipschitz if and only if the derivative f' is bounded.