Sections of Convex Bodies via the Combinatorial Dimension

(Rough notes - no proofs)

These notes are centered at one abstract result in combinatorial geometry, which gives a "coordinate" approach to several classical problems in the asymptotic geometric analysis. Some of these include: the volume ratio method, Elton theorem, Bourgain-Tzafriri theory of restricted invertibility of matrices, Dvoretzky theorem (which is Milman's theorem on euclidean subspaces).

1 Introduction

One of the main aims is to prove the following combinatorial result that compares the volume in $\mathbb{R}^n$ to the *inner volume*, which is the number of the unit cells of $\mathbb{Z}^n$ contained in a set.

Theorem 1 *Let K be a convex body in $\mathbb{R}^n$ with volume at least 1. Then there exists a coordinate projection P in $\mathbb{R}^n$ such that*

$$\underline{\mathrm{vol}}(PK) \geq \mathrm{vol}(cK),$$

where c is a positive absolute constant.

Notice the lack of homogeneity, because the inner volume is taken in a *lower* dimensional subspace than the volume. Usually this automatically implies that P has large rank, and in particular PK contains a cube of side 1. Dualizing, we obtain a large coordinate section of K which is contained in B_1^n, the octahedron with coordinate unit vertices. This method quickly yields, for example, a generalization of the classical volume ratio theorem due to Szarek and Tomczak-Jaegermann.

Theorem 2 (Coordinate Volume Ratio) *Let K be a convex symmetric body in $\mathbb{R}^n$ that contains the unit cube $[-1,1]^n$. Then, for every $1 \le k \le n$, there exists a coordinate subspace E in $\mathbb{R}^n$ of codimension k and such that*

$$K \cap E \subset |cK|^{1/k} \cdot nB_1^n.$$

To deduce the classical volume ratio theorem note a further (random) section of the normalized octahedron nB_1^n is the normalized euclidean ball $\sqrt{n}B_2^n$.

Theorem 1 in particular implies and is equivalent to, the estimate

$$\mathrm{vol}(cK) \le \sum_\sigma \underline{\mathrm{vol}}(P_\sigma K),$$

where the sum is over all subsets σ of $[n]$, and P_σ is the coordinate projection in $\mathbb{R}^n$ onto $\mathbb{R}^\sigma$ (to the empty set we assign the summand 1). The question is, whether the left hand side can be improved to $N(cK, [0,1]^n)$, the number of the unit cubes needed to cover cK. This number is easily seen to be majorized by $\mathrm{vol}(cK)$.

While this problem remains open, we essentially prove this for the normalized euclidean ball instead of the cube $[0,1]$, which seems to enough for current applications that include, in particular, Bourgain-Tzafriri's theory of restricted invertibility of linear operators [BT 87], [BT 91], Elton theorem on l_1-subspaces [E], [T 92], [MV], and, at least one-sided, Dvoretzky theorem for coordinate sections.

Let D be an ellipsoid which contains $[0,1]^n$. The standard volumetric argument gives

$$N(K, D) \le \frac{\mathrm{vol}(K + D)}{\mathrm{vol}(D)} \le \mathrm{vol}(K + D). \tag{1}$$

We prove an asymptotically better general estimate

$$N(K, cD)^{1/2} \le \sum_\sigma \underline{\mathrm{vol}}(P_\sigma K). \tag{2}$$

Remarks. 1. The right hand side of (2) is clearly bounded by

$$\sum_\sigma \mathrm{vol}(P_\sigma K) = \mathrm{vol}(K + [0,1]^n)$$

([Pa] Theorem 1.10). To compare this with (1) note that $[0,1]^n$ can be covered by one translate of D.

2. There is nothing special in the exponent $1/2$ in (2); it can be improved to any constant strictly less than one, but I do not know whether it can equal one.

3. There is much flexibility in D: it can be replaced, for example, by a normalized ball of l_p^n, $0 < p < \infty$; then c needs to be replaced by a constant depending on p. As stated above, it is not known whether (2) is true in the extremal case $p = \infty$, i.e. with $D = [0,1]^n$, but a weaker inequality holds and is discussed in Section 3.

4. The convexity of K plays almost no role in the result; its proof is combinatorial and probabilistic. For example, a weaker notion of convexity (separate convexity, see [M]) would be enough.

The sum in (2) may contain many vanishing terms, as it may happen that $\underline{\mathrm{vol}}(K) = 0$ even though K is nonempty. Let for simplicity K be symmetric with respect to the origin. Then the maximal cardinality of σ in (2) for which the summand is non-zero is at most $d(2K)$, where

$$d(K) = \max\{|\sigma| : \ [-1,1]^\sigma \subset P_\sigma K\}.$$

The number $d(K)$ is called the *combinatorial dimension* of K. This parameter carries over from combinatorics to convexity the classical *Vapnik-Chernovenkis dimension*, defined for discrete sets $A \subset \{-1,1\}^n$ as the maximal cardinality of σ for which $\{-1,1\}^\sigma \subset P_\sigma A$. Inequality (2) can therefore be used to estimate the euclidean entropy of K through the combinatorial dimension of K. This was a problem solved partially in a series of papers of Talagrand (see [T 96], [T 92], [T 02]) and then completely in [MV]. Inequality (2) is, in a sense, a more general result. Its proof, combining probabilistic and combinatorial methods, is built upon [MV]. The weaker result for $D = [0,1]^n$ solves one of the problem of Alon et al. from [ABCH].

By duality, $d(K)$ measures the largest dimension of a coordinate section of K contained in B_1^n, the unit ball of l_1^n (equivalently, B_1^n is the convex hull of $\pm$ the unit vector basis of $\mathbb{R}^n$). Then, as a consequence of (2), we obtain (but this requires a proof) the following generalization of Elton's theorem [E], [T 92], which can be viewed as a *one-sided* Dvoretzky theorem for *coordinate* sections.

Recall that the M-estimate of K is defined as $M_K = \int_{S^{n-1}} \|x\|_K \, d\sigma(x)$, where σ is the normalized Lebesgue measure on the unit euclidean sphere

S^{n-1} and $\|\cdot\|_K$ is Minkowski functional of K.

Theorem 3 *Let K be a symmetric convex body containing B_2^n, the unit euclidean ball in $\mathbb{R}^n$. Let $M = M_K \log^{-1.6}(2/M_K)$. Then there exists a subset σ of $\{1, \dots, n\}$ of size $|\sigma| \geq cM^2 n$, and such that*

$$cM \cdot (K \cap \mathbb{R}^\sigma) \subset \sqrt{|\sigma|} B_1^\sigma. \tag{3}$$

Recall that Dvoretzky theorem guarantees, for $M = M_K$, the existence of a subspace E of dimension $\dim E \geq cM^2 n$ and such that

$$c_1 B_2^n \cap E \subset M \cdot (K \cap E) \subset c_2 B_2^n \cap E.$$

To compare this to (3) note that for a random subpace E in $\mathbb{R}^\sigma$, the section $\sqrt{|\sigma|} B_1^\sigma \cap E$ is equivalent to $B_2^n \cap E$ (with high probability), which therefore recoveres the right hand side of Dvoretzky theorem. Of course, its left hand side could not be recovered in general on coordinate sections, as the example of $K = [-1, 1]^n$ shows.

Considering the identity operator $l_2^n \to (\mathbb{R}^n, \|\cdot\|_K)$ and replacing it by arbitrary operator T, one can view Theorem 3 as a result on the "restricted invertibility" of operators from l_2^n into arbitrary Banach space, generalizing results of Bourgain and Tzafriri [BT 87] proved for operators on l_2^n only. This will be discussed in Section 4.

2 Covering by balls

Definition 4 (see [M]) *A set K in $\mathbb{R}^n$ is called* coordinate convex *if, for every point x and every choice of signs $\theta \in \{-1, 1\}^n$ one can find $y \in K$ such that for all i*

$$\begin{aligned} y(i) &\geq x(i) \quad \textit{if } \theta(i) = 1, \\ y(i) &\leq x(i) \quad \textit{if } \theta(i) = -1. \end{aligned}$$

Clearly, every convex set is coordinate convex; the converse is not true, as shows the example of a cross $\{(x, y) \mid x = 0 \text{ or } y = 0\}$ in $\mathbb{R}^2$.

Next theorem estimates the euclidean entropy of K through the inner Jordan measure of its coordinate projections.

Theorem 5 *There exists an absolute constant c such that the following holds. Let K be a coordinate convex set in $\mathbb{R}^n$, and D be an ellipsoid which contains the cube $[0,c]^n$ and whose axes are colinear to the unit vector basis of $\mathbb{R}^n$. Then*

$$N(K,D)^{1/2} \le \sum_{\sigma} \underline{\mathrm{vol}}(P_\sigma K).$$

Proof. To be included. ■

3 Covering by cubes

Although it is not known whether (2) is true when D is replaced by the cube $[0,1]^n$, this if true if one reduces the exponent $1/2$.

Theorem 6 *There exists an absolute constant c such that the following holds. Let K be a coordinate convex set in $\mathbb{R}^n$, and let $0 < \varepsilon < 1/2$. Then for $D = [0, c/\varepsilon]^n$ we have:*

$$N(K,D)^{1/\lambda} \le \sum_{\sigma} \underline{\mathrm{vol}}(P_\sigma K), \tag{4}$$

where $\lambda = \log^{\varepsilon}(\frac{n}{\log N(K,D)})$.

Let us deduce from this an almost optimal entropy bound for K through the combinatorial dimension of K, a problem studied in particular in [ABCH]. Naturally, the definition of the combinatorial dimension of a general set K needs to allow translations:

$$d(K) = \max\{|\sigma| : \ [-1,1]^\sigma \subset x + P_\sigma K\},$$

where the maximum is over the subsets σ and translates $x \in \mathbb{R}^n$, see [ABCH], [T 02].

Assume that K is a subset of a cube $[0,b]^n$. Then the right hand side of (4) is bounded by

$$\sum_{k=1}^{d(K)} \binom{n}{k} b^k. \tag{5}$$

This can be viewed as a "continuous" version of Sauer-Shelah Lemma, a classical result which states that for a discrete set $A \subset \{-1,1\}^n$ one has

$$N(A,[0,1]^n) = |A| \le \sum_{k=0}^{d(A)} \binom{n}{k}.$$

Since (5) is asymptotically bounded by $(bn/d(K))^{d(K)}$, a straightforward arithmetics gives

Corollary 7 *Under assumptions of Theorem 6, assume also that K is a subset of a cube $[0,b]^n$. Then*

$$\log N(K,D) \lesssim d(K) \cdot \log^{1+\varepsilon}(bn/d(K)). \tag{6}$$

This was proved by Alon et al. [ABCH] with exponent 2 instead of $1+\varepsilon$; it has been a question whether 2 can be reduced.

Inequality (6) is best complemented by the trivial lower bound

$$\log N(K, c_1 D) \ge d(K)$$

for some absolute constant c_1. This means that the entropy of K is nicely controlled by the combinatorial dimension of K.

Proof of Theorem 6. To be included. ■

4 Bourgain-Tzafriri's principle

Theorem 3 can be stated as a restructed invertibility result. In this form, it generalizes Theorem 1.5 [BT 87] proved for operators on l_2^n, and also Theorem 5.2 there.

Recall that the ℓ-norm of $T : l_2^n \to X$ is defined as $\ell(T) = \int_{S^{n-1}} \|Tx\|_X \, d\sigma(x)$.

Theorem 8 *Let X be a Banach space and $T : l_2^n \to X$ be a linear operator with $\ell(T) \ge \sqrt{n}$. Then there exists a subset σ of $\{1,\dots,n\}$ of size*

$$|\sigma| \ge \frac{cn}{\|T\|^2 \log^{3.2} \|T\|}$$

and such that

$$\|Tx\| \ge c \log^{-1.6} \|T\| \cdot \frac{\|x\|_1}{\sqrt{|\sigma|}} \quad \textit{for } x \in \mathbb{R}^\sigma.$$

For $X = l_2^n$ the logarithmic factors can be dropped and the result is the principle restricted invertibility due to Bourgain and Tzafriri ([BT 87] Theorem 1.1). It is not known whether the logarithmic factors can be dropped in general.

Remark. If the space X has type 2, then the ratio $\frac{\|x\|_1}{\sqrt{|\sigma|}}$ can be replaced simply by $c_1\|x\|$, where the constant $c_1 > 0$ depends only on the type 2 constant of X (and actually equals its reciprocal)[1]. This is done by a standard factorization argument using Maurey factorization theorem.

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