The selection problem for bases with brackets and for strong M-bases

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Abstract

We show that every Banach space with a finite-dimensional decomposition has a basis with brackets which is uniformly minimal and such that some its block sequences fail to be strong M-bases. In particular, this shows for every Banach space that the property of a sequence to be strong M-basic is not stable under passing to block-sequences.

1 Main definitions

We recall some standard notions which can be found in [3].

Given sets K in X and V in X^*, we shall use the following notation: $K^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in K\}$ and $V^\top = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in V\}$. A set V in X^* is called *total* if $V^\top = \{0\}$.

The closed linear span of a sequence (or a "system") $\{x_n\}_1^\infty$ of vectors in X is denoted by $[x_n]_1^\infty$. A sequence $\{x_n\}_1^\infty$ is called *minimal* if $x_m \notin [x_n]_{n\neq m}$ for all $m = 1, 2, ...$ One can check easily that $\{x_n\}_1^\infty$ is minimal iff there exists a sequence of *biorthogonal functionals* $\{x_n^*\}_1^\infty \subset X^*$, i.e. so that $x_n^*(x_m) = \delta_{n,m},\ n, m = 1, 2, ...$ (Kronecker's delta). Note that the sequence $\{x_n^*\}$ is uniquely determined iff $\{x_n\}_1^\infty$ is complete in X, i.e. $[x_n]_1^\infty = X$. Next, we say that $\{x_n\}_1^\infty$ is *C-bounded* if $\sup_n \|x_n\| \|x_n^*\| \le C$ for some sequence $\{x_n^*\}_1^\infty$ of biorthogonal functionals. This is clearly equivalent to the following condition: $\inf_m \ \text{dist} \ (x_m/\|x_m\|,\ [x_n]_{n\neq m}) > C^{-1}$. A sequence $\{x_n\}_1^\infty$ is called *uniformly minimal* if it is C-bounded for some C.

A complete minimal sequence $\{x_n\}_1^\infty$ in X is called *M-basis* if the set $\{x_n^*\}_1^\infty$ is total in X^*. Now, $\{x_n\}_1^\infty$ is said to be *M-basic sequence* if it is M-basis in the space $[x_n]_1^\infty$. An M-basic sequence $\{x_n\}_1^\infty$ is called *strong* if $([x_n^*]_{n\in A})^\top \cap [x_n] = [x_n]_{n\notin A}$ for every subset A of $\mathbf{N}$; we also mention a nice equivalent condition [2]: $[x_n]_{n\in A} \cap [x_n]_{n\in B} = [x_n]_{n\in A\cap B}$ for any subsets A and B of $\mathbf{N}$. It was recently shown that every separable Banach space has a uniformly minimal strong M-basis [5].

A sequence $\{x_n\}_1^\infty$ in X is called *basis with brackets* if there exists an increasing sequence $\{r_m\}_1^\infty$ of positive integers such that, setting $r_0 = 0$, we have

$$x = \sum_{m=0}^{\infty} \sum_{n=r_m+1}^{r_{m+1}} x_n^*(x) x_n \qquad \text{for each} \qquad x \in X \quad .$$

Next, $\{x_n\}_1^\infty$ is said to be *basic with brackets sequence* if it is a basis with brackets of the space $[x_n]_1^\infty$. Evidently, every basic with brackets sequence is strong.

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The concept of basis with brackets in a space X is closely connected with the concept of finite-dimensional decomposition (F.D.D.) of X. The sequence $\{X_n\}_1^\infty$ of finite-dimensional subspaces of X is called a *F.D.D.* of X (we write $X = X_1 \oplus X_2 \oplus ...$) if, for each $x \in X$, there exist a unique sequence $\{x_n \in X_n\}_1^\infty$ so that $x = \sum_{n=1}^\infty x_n$. In this case linear projections P_m on X defined by the rule $P_m\left(\sum_{n=1}^\infty x_n\right) = \sum_{n=1}^m x_n$ are bounded and, moreover, $\sup_m \|P_m\| < \infty$. Therefore every basis with brackets $\{x_n\}_1^\infty$ in X determines a F.D.D. $X = [x_n]_{n=r_0+1}^{r_1} \oplus [x_n]_{n=r_1+1}^{r_2} \oplus ...$ Conversely, given a F.D.D. $X = X_1 \oplus X_2 \oplus ...$ and bases $\{x_n\}_{n=r_m+1}^{r_{m+1}}$ of X_m for each m, the sequence $\{x_n\}_1^\infty$ is clearly a basis with brackets of X.

2 Block sequences of bases with brackets

Given a sequence $\{x_n\}_1^\infty$ in a Banach space X and an increasing sequence $\{q_j\}_1^\infty$ of positive integers, we call a *block sequence* of $\{x_n\}_1^\infty$ any sequence $\{y_j\}_1^\infty$ of non-zero vectors with $y_j \in [x_n]_{n=q_j+1}^{q_{j+1}}$, $j = 1, 2, ...$ It is easy to show that every block sequence of a basic sequence, of an M-basic sequence, of a minimal sequence is a basic sequence, an M-basic sequence, a minimal sequence respectively again. Well, what about basic with brackets sequences and strong M-basic sequences?

Definition 2.1 *A strong M-basic sequence is called block strong if every block sequence of it is strong.*

The selection problem is the following question due to A.Plans and A.Reyes: is every strong M-basic sequence block strong? P.Terenzi [4] has constructed a Banach space where the problem has negative answer. The strong M-basic sequence of his example is, moreover, uniformly minimal. Recently, the selection problem has been solved in negative in Hilbert space [1]. We solve the problem for all Banach spaces. Moreover, our strong M-basic sequence is, in fact, a basic with brackets sequence and it can be made uniformly minimal. The example seems to be more simple than the methods of [4] and [1].

Theorem 2.2 *Let $\{e_n\}_1^\infty$ be a basis of a Banach space X such that $\|e_n\| = 1,\ n = 1, 2, ...$ and let $\{\alpha_n\}_1^\infty$, $\{\beta_n\}_1^\infty$ and $\{\gamma_n\}_1^\infty$ be sequences of positive real numbers. By definition, put for each $n = 1, 2, ...$*

$$\begin{aligned} x_{3n-2} &= \alpha_n e_{3n-2} - \gamma_n e_{3n-1}, \\ x_{3n-1} &= \gamma_n e_{3n-1}, \\ x_{3n} &= \gamma_n e_{3n-1} + \beta_n e_{3n}. \end{aligned}$$

Then $\{x_n\}_1^\infty$ is a basis with brackets in X.

Suppose the series $\sum |\alpha_n|$, $\sum |\beta_n|$ and $\sum |\gamma_n|^{-1}$ converge; then the sequence $\{y_n\}_1^\infty$ defined by

$$\begin{aligned} y_{2j-1} &= x_{3j-3} + x_{3j-2}, \\ y_{2j} &= x_{3j-1} \end{aligned}$$

(we assume $x_0 = 0$), is not strong.

Proof**Proof.** Since $[x_{3n-2}, x_{3n-1}, x_{3n}] = [e_{3n-2}, e_{3n-1}, e_{3n}]$ for each n, we see that $\{x_n\}_1^\infty$ is a basis with brackets of X.

Now let us prove that the sequence $\{y_n\}_1^\infty$ is not strong. We define

$$y = \alpha_1 e_1 + \sum_{n=2}^\infty (\beta_{n-1} e_{3n-3} + \alpha_n e_{3n-2}) \quad .$$

It is sufficient to check that the following conditions are satisfied:

1) $y \in [y_j]_1^\infty$;

2) $y \in \left([y_{2j}^*]_1^\infty\right)^\top$, where $y_j^* \in X^*,\ j = 1, 2, ...$ are some biorthogonal functionals for $\{y_j\}$;

3) $y \notin [y_{2j-1}]_1^\infty$.

First, we shall check 1). Pick any functional $x^* \in X^*$ such that $x^*(y_n) = 0,\ n = 1, 2, ...$; it is sufficient to prove that $x^*(y) = 0$. Since $y_{2j} = \gamma_j e_{3j-1}$, we have

$$x^*(e_{3j-1}) = 0,\ j = 1, 2, ... \tag{1}$$

Then it follows from the representation

$$y_1 = \alpha_1 e_1 - \gamma_1 e_2 \tag{2}$$

that

$$x^*(\alpha_1 e_1) = 0. \tag{3}$$

Further, we can write

$$y_{2j-1} = \gamma_{j-1} e_{3j-4} + \beta_{j-1} e_{3j-3} + \alpha_j e_{3j-2} - \gamma_j e_{3j-1},\ j = 2, 3, ... \tag{4}$$

Then, by our assumptions and by (1), we have

$$0 = x^*(y_{2j-1}) = x^*(\beta_{j-1} e_{3j-3} + \alpha_j e_{3j-2}).$$

It now follows from the definition of y and from (3) that $x^*(y) = 0$.

Now, we check 2). Clearly we can choose biorthogonal functionals y_n^* so that $y_{2j}^* = x_{3j-1}^*,\ j = 1, 2, ...$ Since $y = \sum_{n=1}^\infty (\alpha_n e_{3n-2} + \beta_n e_{3n}) = \sum_{n=1}^\infty (x_{3n-2} + x_{3n})$, we obtain for each $j = 1, 2, ...$

$$y_{2j}^*(y) = \sum_{n=1}^\infty x_{3j-1}^*(x_{3n-2} + x_{3n}) = 0 \quad ,$$

so 2) is proved.

It remains to prove 3). Put

$$x^* = \frac{1}{\alpha_1} e_1^* + \sum_{n=2}^\infty \left(\frac{1}{\gamma_{n-1}} e_{3n-4}^* + \alpha_n e_{3n-3}^* - \beta_{n-1} e_{3n-2}^* \right),$$

where e_n^* are the biorthogonal functionals for the basis $\{e_n\}$ (the vector x^* is well defined because $\sup_n \|e_n^*\| < \infty$ and because of the definition of α_n, β_n and γ_n). Obviously, $x^*(y) = 1$. But it follows from (2) and (4) that $x^*(y_{2j-1}) = 0,\ j = 1, 2, ...$ This shows that $y \notin [y_{2j-1}]_1^\infty$. The proof is complete. End Proof ■

Since every Banach space contains a basic sequence, Theorem 2.2 shows that every Banach space has a basic with brackets sequence which is not block strong. Now this will be improved in the following way. We shall see that if a Banach space has a basis with brackets then it has a basis with brackets which is uniformly minimal but not block strong.

Theorem 2.3 *Every Banach space with a F.D.D. has a basis with brackets which is uniformly minimal but not block strong.*

To prove this, two lemmas are required.

Lemma 2.4 *Let X be a finite-dimensional Banach space; let Z be a subspace in X of codimension N and let $\{z_n\}$ be a complete C-bounded minimal system in Z. Then $\{z_n\}$ can be extended to a complete $(\sqrt{N}+1)C$-bounded minimal system in X.*

ProofProof. By the M. Kadets-Snobar theorem, there exists in X^* a linear projection P onto $Z^\perp$ so that $\|P\| \le \sqrt{N}$. Then $(I-P)^*$ is a projection in X onto Z. Let $\{y_n, y_n^*\} \subset X \times X^*$ be an Auerbach basis (i.e. complete 1-bounded minimal system) of the space $\ker(I-P)^*$. It now easily follows that the minimal system $\{z_n\} \cup \{y_n\}$ with biorthogonal functionals $\{(I-P)z_n^*\} \cup \{Py_n^*\}$ satisfies the conditions of the lemma. End Proof ■

Lemma 2.5 *Let $\{g_n\}_1^\infty$ be a C-bounded minimal sequence and let $\{x_n\}_1^N$ be a C'-bounded minimal system such that $[x_n]_1^N = [g_n]_1^N$ and $x^*]_{[g_n]_{N+1}^\infty} = 0$, $n = 1, ..., N$. Then for each integer $K \ge N(C'+1)$, there exists a system $\{\widetilde{x}_n\}_1^K$ which satisfies the following conditions:*

1) $[\widetilde{x}_n]_1^K = [g_n]_1^K$;

2) $\{x_n\}_1^N$ is a block sequence of $\{\widetilde{x}_n\}_1^K$;

3) $\{\widetilde{x}_n\}_1^K$ is a $(\sqrt{N}+1)6C$-bounded minimal system.

ProofProof. We can (and do) assume that $\|g_n\| = \|x_n\| = 1,\ n = 1, 2, ...$ Pick an integer $M \ge 1$ such that $K = NM + N'$ with $0 \le N' < N$. Then

$$M = \frac{K - N'}{N} > \frac{K}{N} - 1 \ge C' = \sup_n \|x_n^*\|, \tag{5}$$

where x_n^* are biorthogonal functionals for x_n.

We are ready to define first NM vectors of a required system. They will be double-indexed, $n = 1, ..., N$ and $j = 1, ..., M$:

$$\widetilde{x}_{nj} = x_n - \frac{1}{M}\left(\sum_{i=N+(n-1)M+1}^{N+nM} g_i\right) + g_{N+(n-1)M+j},$$

$$\widetilde{x}_{nj}^* = \frac{1}{M}x_n^* + g_{N+(n-1)M+j}^*.$$

So, the system $\{\widetilde{x}_n, \widetilde{x}_n^*\}_{n=1}^{NM} := \{\{\widetilde{x}_{nj}, \widetilde{x}_{nj}^*\}_j\}_n$ is defined. It is clear that:

1′) $[\widetilde{x}_n]_1^{NM}$ is a subspace in $[g_n]_1^K$ of codimension N';

2′) $\{x_n\}_1^N$ is a block sequence of $\{\widetilde{x}_n\}_{n=1}^{NM}$: indeed, $x_n = \frac{1}{M}\sum_{j=1}^M \widetilde{x}_{nj}$;

3′) $\{\widetilde{x}_n\}_{n=1}^{NM}$ is $6C$-bounded: it follows from the definition of $\{x_n, x_n^*\}$ and from (5) that $\|\widetilde{x}_{nj}\| \le 3$ and $\left\|\widetilde{x}_{nj}^*\right\| \le 1 + C \le 2C$.

To conclude the proof, it remains to apply Lemma 2.4 to the system $\{z_n\} := \{\widetilde{x}_n\}_{n=1}^{NM}$ in the space $[g_n]_1^K$: we obtain its extension to a $(\sqrt{N'}+1)6C$-bounded system $\{\widetilde{x}_n\}_{n=1}^K$ satisfying 1). Finally, it follows from 2′) that 2) is also true. End Proof ■

Proof of Theorem 2.3. First note that the sequence $\{x_n\}$ in Theorem 2.2 satisfies the following condition. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$and $\{\gamma_n\}$ are fixed. Then there exists a function $F(j, D)$ (where $D = \sup_n \|e_n^*\|$) increasing on D and such that

$$\sup\{\|x_n\|\,\|x_n^*\| : \ n = 3j-2,\ 3j-1,\ 3j\} \le F(j, D), \quad j = 1, 2, ...$$

Now let X be a space with a F.D.D.: $X = X_1 \oplus X_2 \oplus ...$ It can be assumed that $\dim X_j \geq 3\,(F(j,D)+1)$, where D is the decomposition constant of X. Put $q_1 = 0,\ q_{j+1} = q_j + \dim X_j,\ j = 1,2,...$ Then

$$q_{j+1} - q_j \geq 3\,(F(j,D)+1) \tag{6}$$

Let $\{g_n\}_{q_j+1}^{q_{j+1}}$ be an Auerbach basis of X_j. Let us define a sequence $\{e_n\}_1^\infty$ with biorthogonal functionals $\{e_n^*\}_1^\infty$: $e_{3j-k} = g_{q_j+3-k},\ e_{3j-k}^* = g_{q_j+3-k}^*,\ k = 0,1,2,\ j = 1,2,...$ Notice that $\{e_n\}_1^\infty$ is basic sequence in X with $\sup \|e_n^*\| \leq D$; so the system $\{x_n\}_1^\infty$ of the Theorem 2.2 is well defined. The following is true for it:

$$[x_n]_{3j-2}^{3j} = [g_n]_{q_j+1}^{q_j+3} \quad . \tag{7}$$

Apply Lemma 2.5 to the D-bounded system $\{g_n\}_{q_j+1}^{q_{j+1}}$ and to $F(j,D)$-bounded system $\{x_n\}_{3j-2}^{3j}$ with $N = 3$ and with $K = q_{j+1} - q_j$ (the conditions in Lemma are satisfied by (6) and (7)). We obtain a system $\{\widetilde{x}_n\}_{q_j+1}^{q_{j+1}}$ such that:

1) $[\widetilde{x}_n]_{q_j+1}^{q_{j+1}} = [g_n]_{q_j+1}^{q_{j+1}} = X_j$;

2) $\{x_n\}_{3j-2}^{3j}$ is a block sequence of $\{\widetilde{x}_n\}_{q_j+1}^{q_{j+1}}$;

3) $\{\widetilde{x}_n\}_{q_j+1}^{q_{j+1}}$ is $(\sqrt{3}+1)6D$-bounded.

Thus, $\{\widetilde{x}_n\}_1^\infty$ is a basis with brackets which is uniformly bounded and $\{x_n\}_1^\infty$ is its block sequence. By definition, $\{x_n\}_1^\infty$ is not block strong; so $\{\widetilde{x}_n\}_1^\infty$ is not, too. End Proof ■

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