SIZES OF PROJECTIONS OF SYMMETRIC CONVEX BODIES

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Abstract. A sharp inequality is proved for the ℓ-norm of restrictions of a quotient map $Q : X \to F$ from a Banach space X. This extends and improves the "volumetric lemma" of Bourgain, Kalton and Tzafriri. If $E \subset X$ is a subspace, then

$$\ell(Q|_E)\ell(id_{E^*}) \geq \frac{\dim E + \dim F - \dim X}{\kappa(X)},$$

where $\kappa(X)$ is the K-convexity constant of X.

Let X be a finite dimensional Banach space, E be a subspace of X, and $Q : X \to F$ be a quotient map. How does Q act on E? If $\dim E + \dim F \leq \dim X$ then it may happen that E lies competely in the kernel of Q, so that Q vanishes on E. Conversely, if $\dim E + \dim F > \dim X$ then Q can not be zero on E, and we may ask how large $Q|_E$ is. A natural measure of the largeness of $Q|_E$ is the average of $\|Qx\|_F$, where the vector x is uniformly distributed on some fixed euclidean sphere in E. Recall the definition of the ℓ-norm of a linear operator $u : \mathbf{R}^n \to X$:

$$\ell(u) = \Big(\int \|ux\|_X^2 \; d\gamma_n(x) \Big)^{1/2} = \sqrt{n} \Big(\int_{S^{n-1}} \|ux\|_X^2 \; d\sigma(x) \Big)^{1/2},$$

where γ_n is the canonical Gaussian probability measure on $\mathbf{R}^n$, and σ is the normalized Lebesgue measure on the unit euclidean sphere S^{n-1}. (For basic facts about the ℓ-norm see [P] Ch.3, [TJ] §12).

Theorem 1. *Let X be a Banach space,* $\dim X = n$*, Q be a quotient map from X,* $\operatorname{rank} Q = k$*, and let E be a subspace of X,* $\dim E = m$*. Then for any Euclidean structure on E we have*

$$\ell(Q|_E)\ell(id_{E^*}) \geq \kappa(X)^{-1}(k + m - n), \tag{1}$$

where $\kappa(X)$ denotes the K-convexity constant of X.

Remark. It is standard and not difficult to show that $\ell(id_E)\ell(id_{E^*}) \geq m$; however $\ell(Q|_E) \leq \ell(id_E)$.

A variant of (1) was proved by Bourgain, Kalton and Tzafriri in [B-K-T], and was one of the main ingredients in their study of quotients and subspaces of L_p. However, [B-K-T] implied the estimate

$\ell(Q|_E)\ell(id_{E^*}) \geq exp\big(-\frac{cm}{m+k-n}\big)m$ which was rather loose. In contrast, estimate (1) is sharp, as we will discuss after its proof.

Proof of Theorem 1. We will denote the unit euclidean ball in $\mathbf{R}^n$ by B_2^n. Given a symmetric convex body K in a subspace F of l_2^n, we will write $\ell(K)$ instead of $\ell(id : F \to (\mathbf{R}^n, \|\cdot\|_K))$, where $\|\cdot\|_K$ denotes the norm on F induced by K.

The euclidean structure in Theorem 1 is determined by a linear invertible operator $v : l_2^k \to E^*$; the scalar product being taken with respect to the ellipsoid $v(B_2^k)$. First, we will extend this euclidean structure to the whole X^* while keeping $\ell(B_{X^*})$ well bounded. The space E^* is a quotient of X^*; let $R : X^* \to E^*$ be the corresponding quotient map. Fix an $\varepsilon > 0$. By a lemma of G. Pisier ([P] Lemma 9.4 and Remark 9.5) the operator $v : l_2^k \to E^*$ admits a lifting $\widetilde{v} : l_2^k \to X^*$, which satisfies $R\widetilde{v} = v$ and

$$\ell(\widetilde{v}) \leq (1 + \varepsilon/2)\kappa(X)\ell(v). \tag{2}$$

Fix a linear invertible operator $u : l_2^{[k+1,\dots,n]} \to X^*$ satisfying $\ell(u) \leq \frac{\varepsilon}{2}\ell(v)$. We extend the operator $\widetilde{v}$ to an operator $w : l_2^n \to X^*$ by defining $w = \widetilde{v}P_k + u(id - P_k)$, where P_k denotes the coordinate projection onto $\mathbf{R}^k$ in $\mathbf{R}^n$. It follows from (2) that

$$\ell(w) \leq \ell(\widetilde{v}) + \ell(u) \leq (1 + \varepsilon)\kappa(X)\ell(v). \tag{3}$$

Since w is invertible, we can identify X^* with $(\mathbf{R}^n, \|\cdot\|_{X^*})$ so that w is the identity operator in $\mathbf{R}^n$. Moreover, since $v^* = w^*|_E$, the operator v is also the identity in $\mathbf{R}^k$. So E^* is identified with $(\mathbf{R}^k, \|\cdot\|_{E^*})$ and E is identified with $(\mathbf{R}^k, \|\cdot\|_X)$. Then (3) implies that

$$\ell(B_{X^*}) \leq (1 + \varepsilon)\kappa(X)\ell(B_{E^*}). \tag{4}$$

Let $L = \ker Q$. The quotient map Q can be identified with the orthogonal projection in $\mathbf{R}^n$ onto $L^\perp$. The quotient space is then considered as the linear subspace $L^\perp$ with the unit ball $Q(B_X)$. Consider the linear subspace $H = E \cap L^\perp$. Note that

$$\dim H \geq \dim(E) + \dim(L^\perp) - n = k + m - n.$$

In (1), the first factor in the left side equals

$$\begin{aligned} \ell\Big(Q|_E : B_2^k \to Q(B_X)\Big) &= \ell\Big(Q|_E : B_2^n \cap E \to Q(B_X)\Big) \\ &\geq \ell\Big(Q|_H : B_2^n \cap H \to Q(B_X)\Big) \\ &= \ell\Big(Q(B_X) \cap H\Big), \end{aligned} \tag{5}$$

because $Q|_H = id_H$. To estimate (5), we pass to the polar body. Denote by P_H the orthogonal projection onto H in $\mathbf{R}^n$. Then by the standars comparison principle and using (4) we obtain that

$$\begin{aligned} \ell\Big[(Q(B_X) \cap H)^\circ\Big] &= \ell\Big[P_H(B_{X^*} \cap L^\perp)\Big] \\ &\leq \ell(B_{X^*} \cap L^\perp) \leq \ell(B_{X^*}) \\ &\leq (1+\varepsilon)\kappa(X)\ell(B_{E^*}). \end{aligned}$$

Therefore

$$\ell\Big(Q(B_X) \cap H\Big) \geq \frac{\dim H}{\ell\Big[(Q(B_X) \cap H)^\circ\Big]} \geq \frac{k+m-n}{(1+\varepsilon)\kappa(X)\ell(B_{E^*})}.$$

Combining this with (5) and noting that ε is an arbitrary positive number, we complete the proof.

It may seem at the first sight that the right estimate in (1) should be of order $\sqrt{k+m-n}\sqrt{m}$, because the rank of $Q|_E$ is at least $k+m-n$ and the dimension of E is m. However, the following example shows that the estimate in (1) is sharp for all possible k, m, n even if X is a Hilbert space (where $\kappa(X) = 1$).

Example. Let integers k, m, n satisfy $k < n$, $m < n$, $k+m-n > 0$. In the space $X = l_2^n$ there exist an orthogonal projection Q, $\operatorname{rank} Q = k$, a subspace E, $\dim E = m$ and, for any $\varepsilon > 0$, there exists a euclidean structure on E so that

$$\ell(Q|_E)\ell(id_{E^*}) \leq (1+\varepsilon)(k+m-n). \tag{6}$$

Proof. It will be easier to consider the Euclidean structure on E coming from l_2^n, but to let X be a different Hibert space in $\mathbf{R}^n$. So we will now need to define the norm in X.

Let $d = k+m-n$. Let $\mathfrak{Q}, \mathfrak{E}$ be subsets of $\{1, \dots, n\}$ with cardinalities $|\mathfrak{Q}| = k$, $|\mathfrak{E}| = m$, and such that $|\mathfrak{Q} \cap \mathfrak{E}| = d$. Fix a number $\alpha > 0$. Define the norm of x in X, $x = (x_1, \dots, x_n)$ by

$$\|x\|_X^2 = \Big(\sum_{i \in \mathfrak{Q} \triangle \mathfrak{E}} |x_i|^2\Big) + \alpha^2 \Big(\sum_{i \in \mathfrak{Q} \cap \mathfrak{E}} |x_i|^2\Big).$$

Let Q be the coordinate projection onto $\mathbf{R}^{\mathfrak{Q}}$ in $\mathbf{R}^n$. Note that $Q(B_X) = B_X \cap \mathbf{R}^{\mathfrak{Q}}$, so the projection Q in X can indeed be considered as a quotient map. Finally, let $E = \mathbf{R}^{\mathfrak{E}}$. Then

$$\ell(Q|_E) = \ell(B_X \cap \mathbf{R}^{\mathfrak{Q} \cap \mathfrak{E}}) = \sqrt{\alpha^2 |\mathfrak{Q} \cap \mathfrak{E}|} = \sqrt{\alpha^2 d}.$$

Next, the norm of x in E^*, $x = (x_1, \dots, x_n)$, is

$$\|x\|_{E^*}^2 = \Big(\sum_{i \in \mathfrak{Q} \setminus \mathfrak{E}} |x_i|^2 \Big) + \frac{1}{\alpha^2} \Big(\sum_{i \in \mathfrak{Q} \cap \mathfrak{E}} |x_i|^2 \Big).$$

Then

$$\ell(id_{E^*}) = \sqrt{|\mathfrak{E} \setminus \mathfrak{Q}| + \frac{1}{\alpha^2} |\mathfrak{Q} \cap \mathfrak{E}|} = \sqrt{n - k + \frac{1}{\alpha^2} d}.$$

Therefore

$$\ell(Q|_E) \ell(id_{E^*}) = d \sqrt{1 + \alpha^2 \Big(\frac{n-k}{d} \Big)}.$$

So we take α so that $\alpha^2 \ge (\frac{d}{n-k}) \varepsilon$, and (6) follows.

Remark. In Theorem 1, $\ell(Q|_E)$ can also be compared with $\ell(id_E)$. If E is in ℓ-position (see e.g. [G-M]), then it follows that

$$\ell(Q|_E) \ge \kappa(X)^{-2} \Big(\frac{k + m - n}{m} \Big) \ell(id_E).$$

Indeed, it suffices to use the estimate $\ell(id_E) \ell(id_{E^*}) \le \kappa(E) m$ in the ℓ-position of E, and apply (1).

In conclusion, let us state a direct application of Theorem 1 for non-orthogonal projections. If B is a symmetric convex body in $\mathbf{R}^n$, we denote by $\ell(B)$ the ℓ-norm of the formal identity on $(\mathbf{R}^n, \| \cdot \|_B)$.

Corollary 2. *Let K be a symmetric convex body in $\mathbf{R}^n$, and E be a subspace in $\mathbf{R}^n$, $\dim E = k$. Then, for any projection P_E onto E in $\mathbf{R}^n$ (not necessarily orthogonal) and for any euclidean structure on E we have*

$$\ell(P_E(K)) \ell(K^\circ \cap E) \ge \kappa(K)^{-1} (n - 2k).$$

Again, this estimate is sharp, as one can easily see by modifying the example above.

However, we do not know whether the K-convexity of X is essential in our results.

References

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