THE LEAST SINGULAR VALUE OF A RANDOM SQUARE MATRIX IS $O(n^{-1/2})$

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Abstract. Let A be a matrix whose entries are real i.i.d. centered random variables with unit variance and suitable moment assumptions. Then the smallest singular value $s_n(A)$ is of order $n^{-1/2}$ with high probability. The lower estimate of this type was proved recently by the authors; in this note we establish the matching upper estimate.

1. Introduction

Let A be an $n \times n$ matrix whose entries are real i.i.d. centered random variables with suitable moment assumptions. Random matrix theory studies the distribution of the *singular values* $s_k(A)$, which are the eigenvalues of $|A| = \sqrt{A^*A}$ arranged in the non-increasing order. In this paper we study the magnitude of the smallest singular value $s_n(A)$, which can also be viewed as the reciprocal of the spectral norm:

$$s_n(A) = \inf_{x:\ \|x\|_2=1} \|Ax\|_2 = 1/\|A^{-1}\|. \tag{1}$$

Motivated by numerical inversion of large matrices, von Neumann and his associates speculated that

$$s_n(A) \sim n^{-1/2} \quad \text{with high probability.} \tag{2}$$

(See [4], pp. 14, 477, 555). A more precise form of this estimate was conjectured by Smale and proved by Edelman [1] for Gaussian matrices A. For general matrices, conjecture (2) had remained open until we proved in [2] the lower bound $s_n(A) = \Omega(n^{-1/2})$. In the present paper, we shall prove the corresponding upper bound $s_n(A) = O(n^{-1/2})$, thereby completing the proof of (2).

Theorem 1.1 (Fourth moment)**.** *Let A be an $n \times n$ matrix whose entries are i.i.d. centered random variables with unit variance and fourth moment bounded by B. Then, for every $\delta > 0$ there exist $K > 0$ and n_0 which depend (polynomially) only on δ and B, and such that*

$$\mathbb{P}\big(s_n(A) > Kn^{-1/2}\big) \le \delta \qquad \text{for all } n \ge n_0.$$

M.R. was supported by NSF DMS grant 0652684. R.V. was supported by the Alfred P. Sloan Foundation and by NSF DMS grant 0652617.

Remark. The same result but with the reverse estimate, $\mathbb{P}\big(s_n(A) < Kn^{-1/2}\big) \le \delta$, was proved in [2]. Together, these two estimates amount to (2).

Under more restrictive (but still quite general) moment assumptions, Theorem 1.1 takes the following sharper form. Recall that a random variable ξ is called *subgaussian* if its tail is dominated by that of the standard normal random variable: there exists $B > 0$ such that $\mathbb{P}(|\xi| > t) \le 2\exp(-t^2/B^2)$ for all $t > 0$. The minimal B is called the *subgaussian moment* of ξ. The class of subgaussian random variables includes, among others, normal, symmetric ± 1, and in general all bounded random variables.

Theorem 1.2 (Subgaussian)**.** *Let A be an $n \times n$ matrix whose entries are i.i.d. centered random variables with unit variance and subgaussian moment bounded by B. Then for every $K \ge 2$ one has*

$$\mathbb{P}\big(s_n(A) > Kn^{-1/2}\big) \le (C/K)\log K + c^n, \tag{3}$$

where $C > 0$ and $c \in (0,1)$ depend (polynomially) only on B.

Remark. A reverse result was proved in [2]: for every $\varepsilon \ge 0$, one has $\mathbb{P}\big(s_n(A) \le \varepsilon n^{-1/2}\big) \le C\varepsilon + c^n$.

Our argument is an application of the small ball probability bounds and the structure theory developed in [2] and [3]. We shall give a complete proof of Theorem 1.2 only; we leave to the interested reader to modify the argument as in [2] to obtain Theorem 1.1.

2. Proof of Theorem 1.2

By $(e_k)_{k=1}^n$ we denote the canonical basis of the Euclidean space $\mathbb{R}^n$ equipped with the canonical inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\|\cdot\|_2$. By $C, C_1, c, c_1, \dots$ we shall denote positive constants that may possibly depend only on the subgaussian moment B.

Consider vectors $(X_k)_{k=1}^n$ and $(X_k^*)_{k=1}^n$ an n-dimensional Hilbert space H. Recall that the system $(X_k, X_k^*)_{k=1}^n$ is called a *biorthogonal system* in H if $\langle X_j^*, X_k \rangle = \delta_{j,k}$ for all $j,k = 1,\dots,n$. The system is called *complete* if $\mathrm{span}(X_k) = H$. The following notation will be used throughout the paper:

$$H_k := \mathrm{span}(X_i)_{i \ne k}, \qquad H_{j,k} := \mathrm{span}(X_i)_{i \notin \{j,k\}}, \qquad j,k = 1,\dots,n. \tag{4}$$

The next proposition summarizes some elementary and known properties of biorthogonal systems.

Proposition 2.1 (Biorthogonal systems)**.** *1. Let A be an $n \times n$ invertible matrix with columns $X_k = Ae_k$, $k = 1,\dots,n$. Define $X_k^* = (A^{-1})^* e_k$. Then $(X_k, X_k^*)_{k=1}^n$ is a complete biorthogonal system in $\mathbb{R}^n$.*

2. Let $(X_k)_{k=1}^n$ be a linearly independent system in an n-dimensional Hilbert space H. Then there exist unique vectors $(X_k^*)_{k=1}^n$ such that $(X_k, X_k^*)_{k=1}^n$ is a biorthogonal system in H. This system is complete.

3. Let $(X_k, X_k^*)_{k=1}^n$ be a complete biorthogonal system in a Hilbert space H. Then $\|X_k^*\|_2 = 1/\mathrm{dist}(X_k, H_k)$ for $k = 1, \dots, n$. □

Without loss of generality, we can assume that $n \geq 2$ and that A is a.s. invertible (by adding independent normal random variables with small variance to all entries of A).

Let $u, v > 0$. By (1), the following implication holds:

$$\exists x \in \mathbb{R}^n : \ \|x\|_2 \leq u, \ \|A^{-1}x\|_2 \geq vn^{1/2} \quad \text{implies} \quad s_n(A) \leq (u/v)n^{-1/2}. \tag{5}$$

We will now describe how to find such x. Consider the columns $X_k = Ae_k$ of A and the subspaces H_k, $H_{j,k}$ defined in (4). Let P_1 denote the orthogonal projection in $\mathbb{R}^n$ onto H_1. We define the vector

$$x := X_1 - P_1 X_1.$$

Define $X_k^* = (A^{-1})^* e_k$. By Proposition 2.1 $(X_k, X_k^*)_{k=1}^n$ is a complete biorthogonal system in $\mathbb{R}^n$, so

$$\ker(P_1) = \mathrm{span}(X_1^*). \tag{6}$$

Clearly, $\|x\|_2 = \mathrm{dist}(X_1, H_1)$. Conditioning on H_1 and using a standard concentration bound, we obtain

$$\mathbb{P}(\|x\|_2 > u) \leq Ce^{-cu^2}, \qquad u > 0. \tag{7}$$

This settles the first bound in (5) with high probability.

To address the second bound in (5), we write $A^{-1}x = A^{-1}X_1 - A^{-1}P_1X_1 = e_1 - A^{-1}P_1X_1$. Since $P_1X_1 \in H_1$, the vector $A^{-1}P_1X_1$ is supported in $\{2, \dots, n\}$ and hence is orthogonal to e_1. Therefore

$$\begin{aligned} \|A^{-1}x\|_2^2 > \|A^{-1}P_1X_1\|_2^2 &= \sum_{k=1}^n \langle A^{-1}P_1X_1, e_k\rangle^2 \\ &= \sum_{k=1}^n \langle P_1(A^{-1})^* e_k, X_1\rangle^2 = \sum_{k=1}^n \langle P_1 X_k^*, X_1\rangle^2. \end{aligned}$$

The first term of the last sum is zero since $P_1 X_1^* = 0$ by (6). We have proved that

$$\|A^{-1}x\|_2^2 \geq \sum_{k=2}^n \langle Y_k^*, X_1\rangle^2, \quad \text{where} \quad Y_k^* := P_1 X_k^* \in H_1, \qquad k = 2, \dots, n. \tag{8}$$

Lemma 2.2. *$(Y_k^*, X_k)_{k=2}^n$ is a complete biorthogonal system in H_1.*

Proof. By (8) and (6), $Y_k^* - X_k^* \in \ker(P_1) = \mathrm{span}(X_1^*)$, so $Y_k^* = X_k^* - \lambda_k X_1^*$ for some $\lambda_k \in \mathbb{R}$ and all $k = 2, \dots, n$. By the orthogonality of X_1^* to all of X_k, $k = 2, \dots, n$, we have $\langle Y_j^*, X_k \rangle = \langle X_j^*, X_k \rangle = \delta_{j,k}$ for all $j, k = 2, \dots, n$. The biorthogonality is proved. The completeness follows since $\dim(H_1) = n-1$. □

In view of the uniqueness in Part 2 of Proposition 2.1, Lemma 2.2 has the following crucial consequence.

Corollary 2.3. *The system of vectors $(Y_k^*)_{k=2}^n$ is uniquely determined by the system $(X_k)_{k=2}^n$. In particular, the system $(Y_k^*)_{k=2}^n$ and the vector X_1 are statistically independent.* □

By Part 3 of Proposition 2.1, $\|Y_k^*\|_2 = 1/\mathrm{dist}(X_k, H_{1,k})$. We have therefore proved that

$$\|A^{-1}x\|_2^2 \ge \sum_{k=2}^n (a_k/b_k)^2, \text{ where } a_k = \Big|\Big\langle \frac{Y_k^*}{\|Y_k^*\|_2}, X_1 \Big\rangle\Big|, \ b_k = \mathrm{dist}(X_k, H_{1,k}). \tag{9}$$

We will now need to bound a_k above and b_k below. Without loss of generality, we will do this for $k = 2$.

We are going to use a result of [3] that states that random subspaces have no additive structure. The amount of structure is formalized by the concept of the least common denominator. Given parameters $\alpha > 0$ and $\gamma \in (0,1)$, the *least common denominator* of a vector $a \in \mathbb{R}^n$ is defined as

$$\mathrm{LCD}_{\alpha,\gamma}(a) := \inf\big\{\theta > 0 : \ \mathrm{dist}(\theta a, \mathbb{Z}^N) < \min(\gamma\|\theta a\|_2, \alpha)\big\}.$$

The least common denominator of a subspace H in $\mathbb{R}^n$ is then defined as

$$\mathrm{LCD}_{\alpha,\gamma}(H) = \inf\{\mathrm{LCD}_{\alpha,\gamma}(a) : \ a \in H, \|a\|_2 = 1\}.$$

Since $H_{1,2}$ is the span of $n-2$ random vectors with i.i.d. coordinates, Theorem 4.3 of [3] yields that

$$\mathbb{P}\big\{ \mathrm{LCD}_{\alpha,c}((H_{1,2})^\perp) \ge e^{cn} \big\} \ge 1 - e^{-cn}$$

where $\alpha = c\sqrt{n}$, and $c > 0$ is some constant that may only depend on the subgaussian moment B.

On the other hand, note that the random vector X_2 is statistically independent of the subspace $H_{1,2}$. So, conditioning on $H_{1,2}$ and using the standard concentration inequality, we obtain

$$\mathbb{P}\big(b_2 = \mathrm{dist}(X_2, H_{1,2}) \ge t\big) \le Ce^{-ct^2}, \qquad t > 0.$$

Therefore, the event

$$\mathcal{E} := \big\{ \mathrm{LCD}_{\alpha,c}((H_{1,2})^\perp) \ge e^{cn}, \ b_2 < t \big\} \text{ satisfies } \mathbb{P}(\mathcal{E}) \ge 1 - e^{-cn} - Ce^{-ct^2}. \tag{10}$$

Note that the event $\mathcal{E}$ depends only on $(X_j)_{j=2}^n$. So let us fix a realization of $(X_j)_{j=2}^n$ for which $\mathcal{E}$ holds. By Corollary 2.3, the vector Y_2^* is now fixed. By

Lemma 2.2, Y_2^* is orthogonal to $(X_j)_{j=3}^n$. Therefore $Y^* := Y_2^*/\|Y_2^*\|_2 \in (H_{1,2})^\perp$, and because event $\mathcal{E}$ holds, we have

$$\mathrm{LCD}_{\alpha,c}(Y^*) \geq e^{cn}.$$

Let us write in coordinates $a_2 = |\langle Y^*, X_1\rangle| = |\sum_{i=1}^n Y^*(i)X_1(i)|$ and recall that $Y^*(i)$ are fixed coefficients with $\sum_{i=1}^n Y^*(i)^2 = 1$, and $X_1(i)$ are i.i.d. random variables. We can now apply Small Ball Probability Theorem 3.3 of [3] (in dimension $m = 1$) for this random sum. It yields

$$\mathbb{P}_{X_1}(a_2 \leq \varepsilon) \leq C(\varepsilon + 1/\mathrm{LCD}_{\alpha,c}(Y^*) + e^{-c_1 n}) \leq C(\varepsilon + e^{-c_2 n}). \tag{11}$$

Here the subscript in $\mathbb{P}_{X_1}$ means that we the probability is with respect to the random variable X_1 while the other random variables $(X_j)_{j=2}^n$ are fixed; we will use similar notations later.

Now we unfix all random vectors, i.e. work with $\mathbb{P} = \mathbb{P}_{X_1,\dots,X_n}$. We have

$$\begin{aligned}\mathbb{P}(a_2 \leq \varepsilon \text{ or } b_2 \geq t) &= \mathbb{E}_{X_2,\dots,X_n}\mathbb{P}_{X_1}(a_2 \leq \varepsilon \text{ or } b_2 \geq t)\\ &\leq \mathbb{E}_{X_2,\dots,X_n}\mathbf{1}_{\mathcal{E}}\mathbb{P}_{X_1}(a_2 \leq \varepsilon) + \mathbb{P}_{X_2,\dots,X_n}(\mathcal{E}^c)\end{aligned}$$

because $b_2 < t$ on $\mathcal{E}$. By (11) and (10), we continue as

$$\begin{aligned}\mathbb{P}(a_2 \leq \varepsilon \text{ or } b_2 \geq t) &\leq C(\varepsilon + e^{-c_2 n}) + (e^{-cn} + Ce^{-ct^2})\\ &= C_1(\varepsilon + e^{-c_3 t^2} + e^{-cn}) := p(\varepsilon, t, n).\end{aligned}$$

Repeating the above argument for any $k \in \{2, \dots, n\}$ instead of $k = 2$, we conclude that

$$\mathbb{P}\big(a_k/b_k \leq \varepsilon/t\big) \leq p(\varepsilon, t, n) \qquad \text{for } \varepsilon > 0,\ t > 0,\ k = 2, \dots, n. \tag{12}$$

From this we can easily deduce the lower bound on the sum of $(a_k/b_k)^2$, which we need for (9). This can be done using the following elementary observation proved by applying Markov's inequality twice.

Proposition 2.4. *Let $Z_k \geq 0$, $k = 1, \dots, n$, be random variables. Then, for every $\varepsilon > 0$, we have*

$$\mathbb{P}\Big(\frac{1}{n}\sum_{k=1}^n Z_k \leq \varepsilon\Big) \leq \frac{2}{n}\sum_{k=1}^n \mathbb{P}(Z_k \leq 2\varepsilon). \qquad \square$$

We use Proposition 2.4 for $Z_k = (a_k/b_k)^2$, along with the bounds (12). In view of (9), we obtain

$$\mathbb{P}\big(\|A^{-1}x\|_2 \leq (\varepsilon/t)n^{1/2}\big) \leq 2p(4\varepsilon, t, n). \tag{13}$$

Estimates (7) and (13) settle the desired bounds in (5), and therefore we conclude that

$$\begin{aligned}\mathbb{P}\big(s_n(A) \leq (ut/\varepsilon)n^{-1/2}\big) &\geq \mathbb{P}\big(\|x\|_2 \leq u, \|A^{-1}x\|_2 \geq (\varepsilon/t)n^{1/2}\big)\\ &\geq 1 - Ce^{-cu^2} - 2p(4\varepsilon, t, n).\end{aligned}$$

This estimate is valid for all $\varepsilon, u, t > 0$. Choosing $\varepsilon = 1/K$, $u = t = \sqrt{\log K}$, the proof of Theorem 1.2 is complete. □

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