Lecture 1 (March 15)

By $(\varepsilon_i)_{i=1}^\infty$ we denote **Bernoulli (Rademacher)** random variables, i.e. independent, $\{-1,1\}$-valued r.v.'s such that $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$.

A first example of **"exponential inequalities"**:

$$\mathbb{P}\Big\{ \big|\sum_1^N \varepsilon_i\big| > t \Big\} \le 2\exp\Big(-\frac{t^2}{2N}\Big) \quad \text{for all } t>0. \tag{1}$$

More general result - **Bernstein (Hoeffding)** inequality:

$$\mathbb{P}\Big\{ \big|\sum a_i\varepsilon_i\big| > t \Big\} \le 2\exp(-t^2/2) \quad \text{for all } t>0.$$

Exercise 1. *Prove Bernstein inequality with ε_i replaced by independent r.v.'s each uniformly distributed in $[-1,1]$.*

Exercise 2. *It is known that Bernstein inequality is optimal up to constants 2, but the proof is not quite easy. Can one easily prove the optimality of (1) for, say, $t \sim \sqrt{N}$? I.e. there exists an absolute constant C s.t.*

$$\mathbb{P}\Big\{ \big|\sum_1^N \varepsilon_i\big| > C\sqrt{N} \Big\} \le 1/2?$$

Corollary 3. *(another form of Bernstein's inequality). For any finite sequence of numbers (a_i)*

$$\mathbb{P}\Big\{ \big|\sum a_i\varepsilon_i\big| > t \Big\} \le 2\exp\Big(-\frac{t^2}{2\sum a_i^2}\Big) \quad \textit{for all } t>0.$$

The following result allows to measure in different ways the "smallness" of a positive random variable. The result is a part of a folklore.

Proposition 4. *For a positive random variable Z the following are equivalent (T.F.A.E.):*

(i) $\mathbb{E}\exp(Z^2/\alpha^2) \le C$ (for some constants α, C);

(ii) $\mathbb{P}\{Z>t\} \le C'\exp(-t^2/\beta^2)$
(for all $t>0$ and with some constants β, C' independent of t);

(iii) $(\mathbb{E}Z^p)^{1/p} \le C''\sqrt{p}$
(for all $p \ge 1$ and with some constant C'' independent of p).

Exercise 5. *Prove Proposition 4.*

A random variable Z for which $|Z|$ satisfies Proposition 4 is conveniently called a **subgaussian r.v.** A standard gaussian (i.e. $N(0,1)$) random variable is subgaussian.

Bernstein inequality and Proposition 4 imply

Theorem 6. (Khinchine inequality). *For all $0 < p < \infty$ there exist constants A_p and B_p such that*

$$B_p\Big(\sum a_i^2\Big)^{1/2} \le \Big(\mathbb{E}\Big|\sum \varepsilon_i a_i\Big|^p\Big)^{1/p} \le A_p\Big(\sum a_i^2\Big)^{1/2}.$$