LECTURE 10 (MAY 31)

bf Majorizing measure Theorem (MMT).

This is a theorem proved by Fernique and Talagrand. See reference in Lecture 8.

Theorem 1. *Let $(X_t)_{t\in T}$ be a Gaussian process, define the distance d on T as usual. Then*

$$\mathbb{E}\sup_{t\in T} X_t \sim \gamma_2(T,d),$$

where

$$\gamma_2(T,d) = \inf_{\mu}\sup_{t\in T}\int_0^\infty \sqrt{\log\frac{1}{\mu(B(t,\varepsilon))}}\,d\varepsilon.$$

The upper estimate holds for all subgaussian processes.

What we know from the last lecture? ecall that we proved a variant of MMT with partitions:

Proposition 2. *Let $(\mathcal{A}_j)$ be any sequence of partitions of T satisfying the diameter assumption above. If μ is a probability measure on T, then*

$$\mathbb{E}\sup_{t\in T} X_t \le K\sup_{t\in T}\sum_{j>i} r^{-j+1}\sqrt{\log\frac{2}{\mu(A_j(t))}}.$$

We want to show that the right hand side (RHS) here is at most $\gamma_2(T,d)$.

This is done using the so-called partitioning scheme. It requires an equivalent view on MMT - it is NOT through partitions $(\mathcal{A}_j)$, it is NOT through measure μ. It is through construction of what we will call **Talagrand's majorizing functions** φ_j.

A sequence of functions $(\varphi_j)_{j\ge i}$, $\varphi_j : T\to \mathbf{R}^+$ is called Talagrand's majorizing functions if for evey point $s\in T$ and every r^{-j-1}-separated points $t_1,\dots,t_n$ we have

$$\varphi_j \ge r^{-j}\sqrt{\log n} + \min_{l\le n}\varphi_{j+2}(t_l)$$

and

$$S = \sup_{j>i}\sup_{t\in T}\varphi_j(t) < \infty.$$

If one is able to construct such functions for (T,d), then $\mathbb{E}\sup_{t\in T} X_t \leq KS$. On the other hand, the choice

$$\varphi_j(t) = \sup\left\{ \int_0^{r^{-i}} \sqrt{\log \frac{1}{\mu(B(u,\varepsilon))}}\, d\varepsilon, \quad d(t,u) \leq 2r^{-j} \right\}$$

gives one Talagrand majorizing functions satisfying $S = \gamma_2(T,d)$.

In practice it is often easier to construct appropriate Talagrand's majorizing functions on (T,d) than the majorizing measure.