LECTURE 2 (MARCH 22)

A random variable X is called **symmetric** if X has the same distribution as $-X$. Example: Rademacher random variables (ε_i).

Exercise 1. *Prove that if X is symmetric then all the three r.v.'s X, $\varepsilon_1 X$, $\varepsilon_1 |X|$ have the same distribution.*

In the sequel, B is arbitrary Banach space.

Theorem 2. (**Levy's Inequalities.**) *Let (X_i) be independent, symmetric, B-valued r.v.'s Consider the partial sums $S_k = \sum_{i=1}^k X_i$. Then for any N and $t > 0$*

$$\mathbb{P}\{\max_{k \le N} \|S_k\| > t\} \le \mathbb{P}\{\|S_N\| > t\}, \tag{1}$$

$$\mathbb{P}\{\max_{i \le N} \|X_i\| > t\} \le 2\mathbb{P}\{\|S_N\| > t\}. \tag{2}$$

Exercise 3. *Prove (2) following the proof of (1) in your notes.*

Symmetrization.

Let X be a B-valued random variable. Consider its independent copy X'. Then the r.v. $\widetilde{X} = X - X'$ is called the **symmetrization** of X. Its distribution is not far from that of X:

$$\frac{1}{2}\mathbb{P}\{\|X\| > 2\mathbb{E}\|X\| + t\} \le \mathbb{P}\{\|X - X'\| > t\} \le 2\mathbb{P}\{\|X\| > t/2\}.$$

Exercise 4. *Prove that $\mathbb{E}\|X\|^p < \infty$ iff $\mathbb{E}\|\widetilde{X}\|^p < \infty$ for $1 < p < \infty$.*

The crucial fact is taht whenever (X_i) are i.r.v's, and (ε_i) be Rademacher r.v.'s independent from (X_i), then the sequence $(\widetilde{X}_i)$ has the same distribution as $(\varepsilon_i \widetilde{X}_i)$. This yields, for example

Lemma 5. *Let (X_i) be independent, mean zero, B-valued r.v.'s. Let $F : \mathbf{R}_+ \to \mathbf{R}_+$ be a convex function. Then*

$$\mathbb{E}F\Big(\frac{1}{2}\|\sum \varepsilon_i X_i\|\Big) \le \mathbb{E}F\Big(\|\sum X_i\|\Big) \le \mathbb{E}F\Big(2\|\sum \varepsilon_i X_i\|\Big)$$

A natural choice of F is $F(x) = x^p$ with $p > 1$. Then we have a comparison inequality for the p-th moments.

Lemma 5 can be proved using the following

Fact 6. *If F is convex, $\mathbb{E}Y = 0$ then*

$$\mathbb{E}F(\|X + Y\|) \ge \mathbb{E}F(\|X\|).$$

Exercise 7. *Prove the Fact and Lemma for general F, following the proof of the particular case $F(x) = x$ in your notes. Hint: use Jensen's inequality $F(\mathbb{E}Z) \leq \mathbb{E}(FZ)$.*

References

[1] Figiel, Hitczenko, Johnson, Schechtman, Zinn, *Extremal properties of Rademacher functions with applications to the Khintchine and Rosenthal inequalities*, Trans. Amer. Math. Soc. 349 (1997), no. 3, 997–1027.