LECTURE 3 (MARCH 29)

Here is a part of Lecture 2 which was not done the last time:

Theorem 1. (**Contraction Principle.**) *Let $F : \mathbf{R}_+ \to \mathbf{R}_+$ be convex, $(x_i) \in B$. Then for any finite set of real numbers (α_i) s.t. $|\alpha_i| \leq 1$ we have*

$$\mathbb{E}F\Big(\|\sum \alpha_i \varepsilon_i x_i\|\Big) \leq \mathbb{E}F\Big(\|\sum \varepsilon_i x_i\|\Big), \tag{1}$$

$$\mathbb{P}\Big\{\|\sum \alpha_i \varepsilon_i x_i\| > t\Big\} \leq \mathbb{P}\Big\{\|\sum \varepsilon_i x_i\| > t\Big\} \quad \textit{for } t > 0. \tag{2}$$

Generalizations of Khinchine inequality (without proof):

Theorem 2. (**Rosenthal's inequalities** [1].) *Let (X_i) be independent, symmetric. Then for $p \geq 2$*

$$\Big(\mathbb{E}|\sum X_i|^p\Big)^{1/p} \sim \max\Big\{(\sum \mathbb{E}|X_i|^2)^{1/2}, (\sum \mathbb{E}|X_i|^p)^{1/p}\Big\}.$$

Khinchine inequality for a **convex body** K in $\mathbf{R}^n$:

Let $|K|$ denote the volume of K. Define the measure μ_K on K by $\mu_K(A) = \frac{|A \cap K|}{|K|}$. Let $f : K \to \mathbf{R}^n$ be any linear function (then $f(x) = \langle x, y \rangle$ for some $y \in \mathbf{R}^n$). Then the principal result is that all the moments of f are equivalent:

$$(\mathbb{E}|f|^q)^{1/q} \leq (\mathbb{E}|f|^p)^{1/p} \leq c_p(\mathbb{E}|f|^q)^{1/q}, \quad 0 < p < q.$$

Question: for what K is this similar to the usual Khinchine inequality?

This result ia also true if f is a polynomial (Bourgain); the dependence of the constant c_p on p is important; see works of S.Bobkov and O.Guedon for improvements.

Gaussian random variables and vectors

$g_1, \dots, g_N$ denote independent standard Gaussian r.v.'s. Each of them has density $(2\pi)^{-1/2}e^{-x^2/2}\,dx$. $g = (g_1, \dots, g_N) \in \mathbf{R}^N$ is called "standard Gaussian r.vector". It has density $(2\pi)^{-N/2}e^{-\|x\|_2^2/2}\,dx_1 \dots dx_N$. ($\|\cdot\|_2$ denotes the euclidean norm in $\mathbf{R}^N$).

Fact 3. **(Rotation invariance).** *For any orthogonal linear transformation U on $\mathbf{R}^N$ the random vector Ug has the same distribution as g.*

This yields that for any finite set of numbers (a_i) the random vector $\sum a_i g_i$ has the same distribution as $g_1 \cdot (\sum a_i^2)^{1/2}$. This implieas the Khinchine inequality for g_i instead of ε_i.

The euclidean norm of g is STRONGLY concentrated around $\sqrt{N}$:

Fact 4.
$$\mathbb{P}\{1/2 \leq \Big\| \frac{g}{\sqrt{N}} \Big\|_2 \leq 2\} \geq 1 - e^{-cN}.$$

Exercise 5. *Prove the upper bound (Hint: reduce to a general form of Bernstein's inequality, and proceed as in Exercise 1 in Lecture 1).*

Note that $g/\|g\|_2$ is a r.v. uniformly distributed on the euclidean sphere S^{N-1}. Since $\|g\|_2 \sim \sqrt{N}$ most of the time, we have that g is approximately a multiple of a r.v. uniformly distributed on $\S^{N-1}$ by a factor $\sqrt{N}$. Therefore **in many situations the uniform distrbution on the sphere reduces to Gaussian r.v., and vice versa**.

References

[1] Figiel, Hitczenko, Johnson, Schechtman, Zinn, *Extremal properties of Rademacher functions with applications to the Khintchine and Rosenthal inequalities*, Trans. Amer. Math. Soc. 349 (1997), no. 3, 997–1027.