Lecture 4 (April 5)

Fact 1. *If g is the standard Gausiian in $\mathbf{R}^N$ then*

$$c_2\sqrt{\log N} \le \mathbb{E}\|g\|_\infty \le c_1\sqrt{\log N}.$$

Exercise: Prove the lower estimate.

The Fact yields the following comparison between the Gaussian and the Rademacher averages: given vectors $x_1, \dots, x_N$ in a Banach space, we have

$$\mathbb{E}\Big\|\sum_1^N \varepsilon_i x_i\Big\| \le \mathbb{E}\Big\|\sum_1^N g_i x_i\Big\| \le c\sqrt{\log N}\cdot\mathbb{E}\Big\|\sum_1^N \varepsilon_i x_i\Big\|.$$

Martingales

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. An increasing sequence of σ-algebras $\{\emptyset, \Omega\} = \mathcal{A}_o \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}_N = \mathcal{A}$ is called a **filtration**. Let f be $\mathcal{A}$-measurable, or better $f \in L_1(\Omega, \mathcal{A}, \mathbb{P})$. Then the **conditional expectation** of f with respect to $\mathcal{A}_i$, which is denoted by $\mathbb{E}^{\mathcal{A}_i}$, is the (unique) $\mathcal{A}_i$-measurable function f_i satisfying $\int_A f_i \, d\mathbb{P} = \int_A f \, d\mathbb{P}$ for all $\mathcal{A} \in \mathcal{A}_i$. The sequence $f_0, f_1, \dots, f_N$ is called a **martingale**.

Exercise 2. *Find a nontrivial filtration on the symmetric group S_N (the group of permutations on a N-element set.*

Martingale differences are $d_i = f_i = f_{i-1}$.

Properties:

(i) For a $\mathcal{A}_i$-measurable function φ we have $\mathbb{E}^{\mathcal{A}_i}(f\varphi) = \varphi\mathbb{E}^{\mathcal{A}_i}(f)$.

(ii) $\mathbb{E}^{\mathcal{A}_{i-1}} d_i = 0$.

(iii) $\mathbb{E} d_i d_j = 0$ if $i \ne j$.

Theorem 3. *(Bernstein inequality: martingale version)*

$$\mathbb{P}\Big\{|f - \mathbb{E}f| > t\Big\} \le 2\exp(-t^2/2a^2)$$

where $a^2 = \sum \|d_i\|_\infty^2$.

Let X_i be independent Banach-space valued mean zero r.v.'s.

Lemma 4. *(Yurinskii)*

1) $|d_i| \le \|X_i\| + \mathbb{E}\|X_i\|$,

2) $\mathbb{E}^{\mathcal{A}_{i-1}} d_i^2 \le 2\mathbb{E}\|X_i\|$.

And here its application:

$$\mathbb{E}|\|S_N\| - \mathbb{E}\|S_N\||^2 \le \sum \mathbb{E}\|X_i\|^2.$$

Yurinskii's lemma allows to pass from the scalar-valued Bernstein inequality (its martingale version) to a Banach-space valued inequality:

$$\mathbb{P}\Big\{\Big|\|\sum \varepsilon_i x_i\| - \mathbb{E}\|\sum \varepsilon_i x_i\|\Big| > t\Big\} \leq 2\exp(-t^2/2a^2)$$

where $a^2 = \sum \|x_i\|^2$.