LECTURE 5 (APRIL 19)

Here is a part of Lecture 4 which was not done the last time:

Theorem 1. *(Azuma's inequality - a martingale version of Bernstein's inequality)*

$$\mathbb{P}\Big\{|f - \mathbb{E}f| > t\Big\} \le 2\exp(-t^2/2a^2)$$

where $a^2 = \sum \|d_i\|_\infty^2$.

Let X_i be independent Banach-space valued mean zero r.v.'s.

Lemma 2. *(Yurinskii)*
1) $|d_i| \le \|X_i\| + \mathbb{E}\|X_i\|$,
2) $\mathbb{E}^{\mathcal{A}_{i-1}} d_i^2 \le 2\mathbb{E}\|X_i\|$.

And here its application:

$$\mathbb{E}|\|S_N\| - \mathbb{E}\|S_N\||^2 \le \sum \mathbb{E}\|X_i\|^2.$$

Yurinskii's lemma allows to pass from the scalar-valued Bernstein inequality (its martingale version) to a Banach-space valued inequality:

$$\mathbb{P}\Big\{\Big|\|\sum \varepsilon_i x_i\| - \mathbb{E}\|\sum \varepsilon_i x_i\|\Big| > t\Big\} \le 2\exp(-t^2/2a^2)$$

where $a^2 = \sum \|x_i\|^2$.

The same inequality holds for $a = c\mathbb{E}\|\sum \varepsilon_i x_i\|$ (this is without proof). Note that this value of a is in generall not comparable with $(\sum \|x_i\|^2)^{1/2}$. However, this observation easily yields

Theorem 3. *(Kahane inequality) For all* $p > 1$ *and any vectors* x_i *in a Banach space*

$$(\mathbb{E}\|\sum \varepsilon_i x_i\|^p)^{1/p} \sim \mathbb{E}\|\sum \varepsilon_i x_i\|$$

where the constant of the equivalence depends only on p.