Lecture 6 (May 3)

Here we start Gaussian Processes.

A set of random variables $(X_t)_{t\in T}$ is called **random process**. It is *gaussian* if for any finite set (a_i) of scalars, the variable $\sum a_i X_i$ is Gaussian ($i \subset T$), i.e. is $N(0, \sigma^2$ for some σ.

Assume $T = \{1, \dots, N\}$, i.e. $X = (X_1, \dots, X_N)$. The **covariance matrix** is $\Gamma = (\mathbb{E}X_i X_j)$. Γ is (semi)-positive definite. It determines completely the random process.

Exercise 1. *For any set $B \subset \mathbf{R}^n$*

$$\mathbb{P}(X \in B) = \frac{1}{2\pi^{N/2}} \frac{1}{\sqrt{\det \Gamma}} \int_B e^{-\langle \Gamma^{-1} z, z\rangle/2}\, dz.$$

The density of X is

$$f(z_1, \dots, z_N, \Gamma) = \frac{1}{2\pi^{N/2}} \frac{1}{\sqrt{\det \Gamma}} e^{-\langle \Gamma^{-1} z, z\rangle/2}.$$

Exercise 2.

$$\frac{\partial f}{\partial \gamma_{ij}} = \frac{\partial^2 f}{\partial z_i \partial z_j}.$$

Lemma 3. *(Slepian). Let (X_t), (Y_t) be centered Gaussian processes. Assume that*

$$\mathbb{E}|Y_t - Y_s|^2 \le \mathbb{E}|X_t - X_s|^2$$

for all $t, s \in T$. Then

$$\mathbb{E} \max_{t\in T} Y_t \le \mathbb{E} \max_{t \in T} X_t.$$

Entropy. Let (T, d) be a metric space. We define $N(T, d, \varepsilon)$ as the minimal number of balls of radius ε needed to cover T.

Theorem 4. *(Sudakov). Let (X_t) be a centered Gaussian process. Introduce the metrics on T: $d(t, s) = (\mathbb{E}|X_t - X_s|^2)^{1/2}$. Then for $\varepsilon > 0$*

$$\mathbb{E} \sup X_t \ge c\varepsilon \sqrt{\log N(T, d, \varepsilon)}.$$