LECTURE 7 (MAY 10)

Slepian's lemma, which we announced at the end of Lecture 6, was the following.

Lemma 1. *(Slepian's lemma). Let (X_t), (Y_t) be centered Gaussian processes. Assume that*

$$\mathbb{E}|Y_t - Y_s|^2 \leq \mathbb{E}|X_t - X_s|^2$$

for all $t, s \in T$. Then

$$\mathbb{E}\max_{t\in T} Y_t \leq 2\mathbb{E}\max_{t\in T} X_t.$$

Slepian's lemma follows from another result, also due to Slepian, which was proved in Lecture 6:

Lemma 2. *(Slepian). Let $(X_i)_{i\leq N}$, $(Y_i)_{i\leq N}$ be centered Gaussian processes. Assume that*

$$\begin{aligned} &\mathbb{E}X_iX_j \leq \mathbb{E}Y_iY_j && (i \neq j),\\ &\mathbb{E}X_i^2 = \mathbb{E}Y_i^2 && (\forall i). \end{aligned}$$

Then

$$\mathbb{E}\max_{i\leq N} Y_i \leq 2\mathbb{E}\max_{i\leq N} X_i.$$

Sudakov's minoration.

Let (T, d) be a metric space. We define the **entropy** $N(T, d, \varepsilon)$ as the minimal number of balls of radius ε needed to cover T.

The entropy is called also "covering number". It can be related to the "packing number", $N'(T, d, \varepsilon)$, which is defined as a maximal cardinality of an ε-separated set T_0 in T (the latter means that $d(t, s) > \varepsilon$ for $t, s, \in T_0$. The following relation holds:

$$N(T, d, \varepsilon) \leq N'(T, d, \varepsilon/2)$$

and in many the two numbers are actually equivalent.

Theorem 3. *(Sudakov's minoration). Let (X_t) be a centered Gaussian process. Introduce the metrics on T: $d(t, s) = (\mathbb{E}|X_t - X_s|^2)^{1/2}$. Then for $\varepsilon > 0$*

$$\mathbb{E}\sup X_t \geq c\varepsilon\sqrt{\log N(T, d, \varepsilon)}.$$

This theorem follows from Slepian's lemma.

Example. Let $(\mathbf{R}^n, \|\cdot\|)$ be a normed space with the unit ball $K = \{x \in \mathbf{R}^n : \ \|x\| = 1\}$. I.e. K is arbitrary convex symmetric body in $\mathbf{R}^n$. By K° we denote the *polar* of K, i.e. $K^\circ = \{x \in \mathbf{R}^n : \langle x, y\rangle \leq 1 \forall y \in K\}$. Then for the standard Gaussian g in $\mathbf{R}^n$

$$\mathbb{E}\|g\| \geq \varepsilon \sqrt{\log N(K^\circ, B_2^n)}$$

where by $N(K, B)$ we naturally denote the minimal number of translates of B needed to cover K.

This estimate follows from Sudakov's minoration for the Gaussian process $X = (X_t)_{t \in K^\circ}$, $X_t = \langle g, t\rangle$.